

THE AUTOMORPHISMS OF THE HOLOMORPH OF A FINITE ABELIAN GROUP

BY

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In 1908 G. A. Miller [4] proved that the holomorph of a finite abelian group of odd order has only inner automorphisms. The group of automorphisms of the holomorph of an arbitrary group was studied by Gol'fand [3] who found some cases in which the outer automorphism group has order one or two. In the present paper I determine explicitly the outer automorphism group Θ of the holomorph H of an arbitrary finite abelian group G . If G is the direct product of a group of odd order, a group of order two, and a cyclic group of order 2^n where $n \geq 2$, then Θ is the direct product of a finite number of groups of order two and a non-abelian group Θ^* of order six or eight. If $n=2$ then Θ^* is isomorphic to the symmetric group of order six, and if $n \geq 3$ then Θ^* is the octic group. In all other cases Θ is either trivial or the direct product of a finite number of groups of order two.

Let A be the group of all automorphisms of the finite abelian group G , let $\mathfrak{B}$ be the group of all automorphisms of H that map G onto itself, and let $\mathfrak{I}$ be the group of all inner automorphisms of H . Then $\mathfrak{B}/\mathfrak{I}$ can be identified with the first cohomology group $H^1(A, G)$. Thus $H^1(A, G)$ can be regarded as a subgroup of Θ . Now G is an invariant subgroup of H , and it is known [6] that H has at most four invariant subgroups isomorphic to G . It follows that $H^1(A, G)$ has index at most four in Θ .

In Part I the first cohomology group $H^1(A, G)$ is determined explicitly—it is either trivial or the direct product of groups of order two. In Part II the results of Part I are combined with the results of [6] to determine Θ explicitly.

I. $H^1(A, G)$

1. The role of $H^1(A, G)$. Let G be a finite abelian group and A its group of automorphisms. The holomorph H of G is defined⁽¹⁾ to be the semi-direct product of A and G . Thus H is the group of all ordered pairs (g, σ) , $g \in G$, $\sigma \in A$, with multiplication given by

$$(g, \sigma)(a, \tau) = (g\sigma a, \sigma\tau).$$

The first eight lower case Roman letters will be used to denote group

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⁽¹⁾ There are two other well known definitions of the holomorph. It has been defined as the group of one-to-one mappings of G onto itself that is generated by the automorphisms of G and the left multiplications $g \rightarrow ag$. Suppose G is represented as a regular permutation group on n letters. Then, according to the original definition, the holomorph is the normalizer of G in the symmetric group S_n . These three definitions are equivalent up to isomorphism.

elements (usually elements of G), and the remaining ones to denote non-negative integers. Lower case Greek letters will be used to denote homomorphisms of groups (usually automorphisms of G), and capital Greek letters will be used to denote automorphisms of the holomorph H and one dimensional cocycles.

We denote the identity of G by e , and that of A by ϵ .

The elements of the form (g, ϵ) form an invariant subgroup of H , the mapping $g \rightarrow (g, \epsilon)$ is an imbedding of G in H , and we will henceforth identify the element g in G with the element (g, ϵ) in H . On the other hand we will distinguish carefully between the element σ in A and the element (e, σ) in H .

Let $I_{(g, \sigma)}$ denote the inner automorphism of H corresponding to (g, σ) . Thus

$$I_{(g, \sigma)}(a, \tau) = (g, \sigma)(a, \tau)(g, \sigma)^{-1}.$$

In particular $I_{(e, \sigma)}g = \sigma g$, so that every automorphism of G can be extended to an inner automorphism of H .

Let $\mathfrak{A}$ be the group of all automorphisms of H , and let $\mathfrak{J}$ be the group of all inner automorphisms of H . Let $\mathfrak{B}$ be the group of all automorphisms of H that map G onto itself, and let $\mathfrak{C}$ be the group of all automorphisms of H that act as the identity on G . Then $\mathfrak{A} \supseteq \mathfrak{B} \supseteq \mathfrak{C}$ and $\mathfrak{J}$ is an invariant subgroup of both $\mathfrak{A}$ and $\mathfrak{B}$. Let $\mathfrak{O} = \mathfrak{A}/\mathfrak{J}$ be the outer automorphism group of H . Our ultimate goal is the determination of $\mathfrak{O}$, and we will begin with the study of $\mathfrak{B}/\mathfrak{J}$.

Suppose $\Omega \in \mathfrak{B}$. The restriction of Ω to G is an automorphism σ of G , and we have $\Omega I_{(e, \sigma)}^{-1}g = g$ for all $g \in G$. Hence $\Omega I_{(e, \sigma)}^{-1} \in \mathfrak{C}$. It follows that $\mathfrak{B} = \mathfrak{C}\mathfrak{J}$.

Suppose $\Gamma \in \mathfrak{C}$. Let Γ' be the mapping of A into G and $\sigma \rightarrow \bar{\sigma}$ the mapping of A into itself such that $\Gamma(e, \sigma) = (\Gamma'\sigma, \bar{\sigma})$ for all $\sigma \in A$. If g is an arbitrary element of G we have $(e, \sigma)g = \sigma g(e, \sigma)$,

$$\Gamma((e, \sigma)g) = (\Gamma'\sigma, \bar{\sigma})g = \bar{\sigma}g(\Gamma'\sigma, \bar{\sigma}),$$

and

$$\Gamma(\sigma g(e, \sigma)) = \sigma g(\Gamma'\sigma, \bar{\sigma}).$$

Hence $\bar{\sigma}g = \sigma g$ for all $g \in G$. Therefore $\bar{\sigma} = \sigma$. Furthermore $(\Gamma'\sigma, \sigma)(\Gamma'\tau, \tau) = \Gamma((e, \sigma)(e, \tau)) = \Gamma(e, \sigma\tau) = (\Gamma'(\sigma\tau), \sigma\tau)$. Comparing first components we obtain

$$(1) \quad (\Gamma'\sigma)(\sigma\Gamma'\tau) = \Gamma'(\sigma\tau)$$

for all σ, τ in A . We note that (1) is the condition that Γ' be an element of $Z^1(A, G)$, the group of one dimensional cocycles or crossed homomorphisms of A into G . Conversely if $\Gamma' \in Z^1(A, G)$, then the mapping Γ given by

$$(2) \quad \Gamma(g, \sigma) = (g\Gamma'\sigma, \sigma)$$

is an automorphism of H and hence an element of $\mathfrak{C}$. The mapping $\Gamma \rightarrow \Gamma'$ is

an isomorphism of $\mathfrak{C}$ onto $Z^1(A, G)$. Henceforth we will identify the cocycle Γ' with the element Γ of $\mathfrak{C}$ given by (2). Then we have $\mathfrak{C} = Z^1(A, G)$.

Suppose now that $I_{(a, \tau)} \in \mathfrak{C}$. Then

$$g = I_{(a, \tau)}g = I_a I_{(e, \tau)}g = \tau g$$

for all $g \in G$. Hence $\tau = e$ and $(a, \tau) = a$. Furthermore

$$I_a(e, \sigma) = a(e, \sigma)a^{-1} = (a\sigma a^{-1}, \sigma).$$

It follows that if $\Gamma \in Z^1(A, G)$, then the condition that Γ be an inner automorphism of H is that it be of the form Γ_a , where

$$(3) \quad \Gamma_a \sigma = a\sigma a^{-1},$$

a a fixed element of G . Now this is the condition that Γ be an element of $B^1(A, G)$, the group of one dimensional coboundaries or splitting homomorphisms of A into G . Thus $\mathfrak{g} \cap \mathfrak{C} = B^1(A, G)$ and we have

$$\mathfrak{g}/\mathfrak{g} = \mathfrak{C}\mathfrak{g}/\mathfrak{g} \cong \mathfrak{C}/\mathfrak{g} \cap \mathfrak{C} = Z^1(A, G)/B^1(A, G) = H^1(A, G),$$

the first cohomology group of A acting on G . The first cohomology group $H^1(A, G)$ is thus isomorphic to $\mathfrak{g}/\mathfrak{g}$ under the natural isomorphism

$$\Gamma B^1(A, G) \rightarrow \Gamma \mathfrak{g}.$$

We are now faced with the problem of explicit determination of $H^1(A, G)$ to which we devote the remainder of part I.

2. Additional notation and preliminary lemmas. We will write

$$(4) \quad Q = Q_1 \times Q_2 \times \cdots \times Q_m,$$

or $Q = \prod Q_i$, if the Q_i are subgroups of the group Q , and Q is the direct product of the Q_i . Suppose (4) holds, let S be a subset of $\{1, 2, \dots, m\}$, and put

$$Q_S = \prod_{i \in S} Q_i.$$

If σ is any automorphism of Q_S we identify σ with the automorphism σ' of Q such that

$$\sigma'g = \begin{cases} \sigma g & \text{if } g \in Q_S, \\ g & \text{if } g \in Q_j, j \notin S. \end{cases}$$

This identification does not depend on Q_S and Q alone, but also on the groups $Q_j, j \notin S$. However we will not have occasion to deal with two decompositions of the same group simultaneously, except in cases where they have a common refinement. Hence we can identify σ with σ' without danger of ambiguity, and we will do so freely.

We need to know a set of generators for the group of automorphisms of G . It is sufficient to settle this question for the prime power case, which is treated in the following lemma.

LEMMA 1. Let $G^{(p)}$ be a finite abelian group whose order is a power of the prime number p . Let $G^{(p)} = D_1 \times D_2 \times \cdots \times D_l$, where D_i is cyclic of order m_i and $m_1 \geq m_2 \geq \cdots \geq m_l > 1$. Let d_i be a generator of D_i , and let A_i be the group of automorphisms of D_i . For $1 \leq j \leq l-1$, let $s_j = m_j/m_{j+1}$, and let γ_j and δ_j be the automorphisms of $D_j \times D_{j+1}$ such that

$$\begin{aligned}\gamma_j d_j &= d_j d_{j+1}, & \gamma_j d_{j+1} &= d_{j+1}, \\ \delta_j d_j &= d_j, & \delta_j d_{j+1} &= d_j^{s_j} d_{j+1}.\end{aligned}$$

Then the group of all automorphisms of $G^{(p)}$ is generated by the automorphisms $\gamma_j, \delta_j, 1 \leq j < l$, and the groups $A_i, 1 \leq i \leq l$.

Proof by induction on l . Lemma 1 is trivial if $l=1$. Suppose that $l>1$ and that Lemma 1 holds for the subgroup $\tilde{G} = D_2 \times D_3 \times \cdots \times D_l$. Let $\tilde{A}$ be the group generated by the automorphisms $\gamma_j, \delta_j, 1 \leq j < l$, and the groups $A_i, 1 \leq i \leq l$. By the induction hypothesis any automorphism ω of $\tilde{G}$ belongs to $\tilde{A}$. Let σ be an arbitrary automorphism of $G^{(p)}$. We write

$$\sigma d_1 = \prod_{i=1}^l d_i^{u_i}.$$

We begin by constructing an automorphism $\psi \in \tilde{A}$ such that $\psi d_1 = \sigma d_1$. There are two possibilities to be considered: (I) $p \mid u_1$. Here for some $j \geq 2$ we must have $p \nmid u_j$ and $m_1 = m_j$. Therefore $m_1 = m_2, s_1 = 1$, and there is an automorphism ρ of $\tilde{G}$ such that

$$\rho d_2 = \prod_{i=2}^l d_i^{u_i}.$$

For such a ρ we have

$$\rho \delta_1^{u_1-1} \gamma_1 d_1 = \rho \delta_1^{u_1-1} (d_1 d_2) = \rho (d_1^{u_1} d_2) = \sigma d_1,$$

and we put $\psi = \rho \delta_1^{u_1-1} \gamma_1$, which belongs to $\tilde{A}$. (II) $p \nmid u_1$. In this case there exists an automorphism τ of D_1 such that $\tau d_1 = d_1^{u_1}$ and an automorphism ρ' of $\tilde{G}$ such that

$$\rho' d_2 = d_2 \prod_{i=3}^l d_i^{u_i},$$

where ρ' is understood to be ϵ if $l=2$. Then

$$\begin{aligned}\tau \gamma_1^{u_1-1} \rho' \gamma_1 d_1 &= \tau \gamma_1^{u_1-1} \rho' (d_1 d_2) = \tau \gamma_1^{u_1-1} \left(d_1 d_2 \prod_{i=3}^l d_i^{u_i} \right) \\ &= \tau \left(d_1 \prod_{i=2}^l d_i^{u_i} \right) = \sigma d_1,\end{aligned}$$

and we put $\psi = \tau \gamma \rho_1^{u_1-1} \gamma_1$, which is an element of $\tilde{A}$.

Thus in both cases we have found an automorphism $\psi \in \tilde{A}$ such that $\psi d_1 = \sigma d_1$. Put $\sigma_1 = \psi^{-1}\sigma$. Then $\sigma_1 d_1 = d_1$. We now write

$$\sigma_1 d_j = \prod_{i=1}^l d_i^{v_{ij}}, \quad 2 \leq j \leq l.$$

Let ψ_1 be the endomorphism of $\tilde{G}$ given by

$$\psi_1 d_j = \prod_{i=2}^l d_i^{v_{ij}}, \quad 2 \leq j \leq l.$$

Since $\sigma_1 d_1 = d_1$ it follows that ψ_1 has a trivial kernel. Therefore ψ_1 is an automorphism of $\tilde{G}$. Hence $\psi_1 \in \tilde{A}$ and we put $\sigma_2 = \psi_1^{-1}\sigma_1$. Then $\sigma_2 d_1 = d_1$ and

$$\sigma_2 d_j = d_1^{v_{1j}} d_j, \quad 2 \leq j \leq l.$$

Now put $v_j = v_{1j} m_j / m_1$. Since $(\sigma_2 d_j)^{m_j} = e$ it follows that v_j is an integer. For $j \geq 3$ let τ_j be the automorphism of $D_2 \times D_j$ such that

$$\tau_j d_2 = d_2^{-1}, \quad \tau_j d_j = d_2^{m_2/m_j} d_j.$$

Then $(\tau_j \delta_1)^2 d_j = d_1^{m_1/m_j} d_j$ and $(\tau_j \delta_1)^2 d_i = d_i$ for $i \neq j$. It follows that

$$\sigma_2 = \delta_1^{v_2} \prod_{j=3}^l (\tau_j \delta_1)^{2v_j} \in \tilde{A}.$$

Therefore $\sigma = \psi \psi_1 \sigma_2 \in \tilde{A}$. This completes the proof of Lemma 1.

We write the finite abelian group G in the form $G = G' \times G^{(2)}$, where G' has odd order, and the order of $G^{(2)}$ is a power of 2. Let A' and $A^{(2)}$ be the groups of automorphisms of G' and $G^{(2)}$ respectively. Then $A = A' \times A^{(2)}$. We write

$$G^{(2)} = C_1 \times C_2 \times \cdots \times C_k,$$

where C_i is cyclic of order n_i , $n_1 \geq n_2 \geq \cdots \geq n_k \geq 2$, and each of these n_i is a power of 2. Let c_i be a generator of C_i . For $j > k$ we let C_j be the trivial group, $c_j = e$, and $n_j = 1$. Furthermore we put $r_i = n_i / n_{i+1}$ which must be a non-negative power of 2. If G has odd order, then $G^{(2)}$ is trivial and $k = 0$. We will use the decomposition

$$G = G' \times C_1 \times \cdots \times C_k$$

to identify automorphisms of such groups as G' , $G^{(2)}$, C_i , $C_i \times C_{i+1}$ with elements of A . Let λ , λ' , and λ'' be the automorphisms of G , G' , and $G^{(2)}$ respectively such that

$$\lambda g = g^{-1}, \quad \lambda' g' = g'^{-1}, \quad \lambda'' g'' = g''^{-1}$$

for all $g \in G$, $g' \in G'$, $g'' \in G^{(2)}$. Clearly $\lambda = \lambda' \lambda''$.

Now let λ_i and ξ_i be the automorphisms of C_i such that

$$\lambda_i c_i = c_i^{-1}, \quad \xi_i c_i = c_i^5.$$

Let η_i and θ_i be the automorphisms of $C_i \times C_{i+1}$ such that

$$\begin{aligned} \eta_i c_i &= c_i c_{i+1}, & \eta_i c_{i+1} &= c_{i+1}, \\ \theta_i c_i &= c_i, & \theta_i c_{i+1} &= c_i^{r_i} c_{i+1}. \end{aligned}$$

We note that λ_i and ξ_i generate the group of automorphisms of C_i . Hence, according to Lemma 1, the automorphisms $\lambda_i, \xi_i, 1 \leq i \leq k$, and $\eta_j, \theta_j, 1 \leq j < k$, generate $A^{(2)}$. It is clear that $\lambda_i = \xi_i = \epsilon$ for $i > k$, and $\eta_j = \theta_j = \epsilon$ for $j \geq k$.

Let J be the center of H . Then $(a, \sigma) \in J$ if and only if

$$(a, \sigma) = g(a, \sigma)g^{-1} = (ag\sigma g^{-1}, \sigma)$$

and

$$(a, \sigma) = (e, \tau)(a, \sigma)(e, \tau)^{-1} = (\tau a, \tau \sigma \tau^{-1})$$

for all $g \in G, \tau \in A$. It follows that $(a, \sigma) \in J$ if and only if $\sigma = \epsilon$ and $\tau a = a$ for all $\tau \in A$. Thus we have the following result:

LEMMA 2. *The center J of H is the group of all characteristic elements of G .*

LEMMA 3. *If G has a nontrivial characteristic element h , then $n_1 > n_2$ and $h = c_1^{n_1/2}$. Conversely if $n_1 > n_2$, then $c_1^{n_1/2}$ is a characteristic element of G .*

Proof. Let h be a nontrivial characteristic element of G . We have $h = \lambda h = h^{-1}$ and hence h has order 2. Therefore $h \in G^{(2)}$ and $n_1 \geq 2$. If $h \neq c_1^{n_1/2}$ then there is an automorphism τ of $G^{(2)}$ such that $\tau h = h c_1^{n_1/2} \neq h$. Thus $h = c_1^{n_1/2}$. If $n_1 = n_2$ then interchanging the roles of C_1 and C_2 we obtain $h = c_2^{n_2/2}$, a contradiction. Therefore $n_1 > n_2$.

Conversely if $n_1 > n_2$, then $c_1^{n_1/2}$ is the only element of G of order 2 of the form $g^{n_1/2}, g \in G$. Thus $n_1 > n_2$ implies that $c_1^{n_1/2}$ is a characteristic element of G .

Combining Lemmas 2 and 3 we obtain J explicitly:

LEMMA 4. *The center J of H is trivial if $n_1 = n_2$. If $n_1 > n_2$ then J has order two and is generated by $c_1^{n_1/2}$.*

Now put

$$N_j = G' \times \prod_{i \neq j} C_i, \quad N_{j,j'} = G' \times \prod_{i \neq j, j'} C_i.$$

Let $J', J^{(2)}, K_j$, and $K_{j,j'}$ be the groups of characteristic elements of $G', G^{(2)}, N_j$, and $N_{j,j'}$ respectively. Applying Lemma 3 to these various groups we obtain the following information:

LEMMA 5. *J' is trivial and $J^{(2)} = J$. If $3 \leq j < j'$, then $K_j = K_{j,j'} = J$. If*

$2 \leq j < j'$, then K_j and $K_{j,j'}$ are subgroups of C_1 of order at most two. If $2 \leq j < j'$ and $n_1 > n_2$, then $K_j = K_{j,j'} = J$. Furthermore $K_1 \subseteq C_2$ and $K_{1,2} \subseteq C_3$. If $n_2 = n_3$ then K_1 is trivial, and if $n_2 > n_3$ then K_1 has order two. If $n_3 = n_4$ then $K_{1,2}$ is trivial, and if $n_3 > n_4$ then $K_{1,2}$ has order two.

LEMMA 6. Suppose that $G = Q_1 \times Q_2$ and that $\bar{J}$ is the group of characteristic elements of Q_1 . Let τ be an automorphism of Q_2 . If $\Gamma \in Z^1(A, G)$, then $\Gamma\tau \in \bar{J} \times Q_2$ and $\Gamma(\tau^2) \in Q_2$. Furthermore if $\bar{Z}$ is the group of all cocycles $\bar{\Gamma}$ such that $\bar{\Gamma}\tau \in Q_2$, then $[Z^1(A, G) : \bar{Z}]$, the index of $\bar{Z}$ in $Z^1(A, G)$, is at most two.

Proof. Let σ be any automorphism of Q_1 . Then $\sigma\tau = \tau\sigma$ and hence

$$(\Gamma\sigma)(\sigma\Gamma\tau) = (\Gamma\tau)(\tau\Gamma\sigma).$$

Since G is abelian this can be written

$$(\Gamma\tau)^{-1}(\sigma\Gamma\tau) = (\Gamma\sigma)^{-1}(\tau\Gamma\sigma).$$

Now $g^{-1}\sigma g \in Q_1$ and $g^{-1}\tau g \in Q_2$ for all $g \in G$. Therefore $(\Gamma\tau)^{-1}(\sigma\Gamma\tau) \in Q_1 \cap Q_2$, which consists of e alone. Hence $\sigma\Gamma\tau = \Gamma\tau$. Since this holds for any automorphism σ of Q_1 we must have $\Gamma\tau \in \bar{J} \times Q_2$. We may write $\Gamma\tau = \bar{h}g_2$, $\bar{h} \in \bar{J}$, $g_2 \in Q_2$. Now $\bar{J}$ is the group of characteristic elements of a finite abelian group. Therefore, by Lemma 3, $\bar{J}$ has order 1 or 2. Hence $\bar{h}\tau\bar{h} = \bar{h}^2 = e$ for any $\bar{h} \in \bar{J}$. It follows that $\Gamma(\tau^2) = (\Gamma\tau)(\tau\Gamma\tau) = g_2\tau g_2 \in Q_2$.

Finally we note that the mapping $\Gamma \rightarrow \bar{h}$ is a homomorphism of $Z^1(A, G)$ into $\bar{J}$ with kernel $\bar{Z}$. Hence $[Z^1(A, G) : \bar{Z}]$ is at most the order of $\bar{J}$, which in turn is at most 2.

LEMMA 7. If $G = Q_1 \times Q_2$, if τ and $\bar{\lambda}$ are automorphisms of Q_2 , and if $\bar{\lambda}g_2 = g_2^{-1}$ for all $g_2 \in Q_2$, then

$$(5) \quad (\Gamma\tau)^2 = (\Gamma\bar{\lambda})(\tau\Gamma\bar{\lambda})^{-1}$$

for all $\Gamma \in Z^1(A, G)$.

Proof. As in Lemma 6 let $\bar{J}$ be the group of characteristic elements of Q_1 . Since $\bar{J}$ has order one or two it follows that $\bar{\lambda}\bar{h} = \bar{h} = \bar{h}^{-1}$ for all $\bar{h} \in \bar{J}$. Hence $\bar{\lambda}g = g^{-1}$ for all $g \in \bar{J} \times Q_2$. We have $\Gamma\tau \in \bar{J} \times Q_2$ by Lemma 6, and therefore $\bar{\lambda}\Gamma\tau = (\Gamma\tau)^{-1}$. Now $\bar{\lambda}$ is in the center of the group of automorphisms of Q_2 . Therefore $\tau\bar{\lambda} = \bar{\lambda}\tau$,

$$(\Gamma\tau)(\tau\Gamma\bar{\lambda}) = \Gamma\bar{\lambda}(\bar{\lambda}\Gamma\tau) = (\Gamma\bar{\lambda})(\Gamma\tau)^{-1},$$

and (5) follows at once.

If we apply Lemma 7 to the case $Q_2 = G$, Q_1 trivial, then we obtain

$$(6) \quad \Gamma^2\tau = (\Gamma\tau)^2 = (\Gamma\bar{\lambda})(\tau\Gamma\bar{\lambda})^{-1}$$

for all $\tau \in A$, $\Gamma \in Z^1(A, G)$. It follows from (6) that the square of every ele-

ment of $Z^1(A, G)$ is a coboundary. Therefore *every nontrivial element of $H^1(A, G)$ has order two.*

LEMMA 8. *If $n_1 \geq 8$, then $(\Gamma\lambda_1)^{n_1/2} = e$ for all $\Gamma \in Z^1(A, G)$.*

Proof. Let τ be the automorphism of C_1 such that $\tau c_1 = c_1^{1+n_1/2}$. Then $\tau^2 = e$. By Lemma 6, $\Gamma\lambda_1$ and $\Gamma\tau$ are elements of $C_1 \times K_1$. For any $g \in C_1 \times K_1$ we have $\tau g = g^{1+n_1/2}$. It follows that $\tau\Gamma\lambda_1 = (\Gamma\lambda_1)^{1+n_1/2}$ and

$$e = \Gamma(\tau^2) = (\Gamma\tau)(\tau\Gamma\tau) = (\Gamma\tau)^{2+n_1/2}.$$

Now n_1 is a power of 2 and $n_1 \geq 8$. Hence $2+n_1/2$ is not divisible by 4. Furthermore $\Gamma\tau \in G^{(2)}$ so that the order of $\Gamma\tau$ is a power of 2. Hence $(\Gamma\tau)^2 = e$. Applying Lemma 7 to the case $Q_1 = N_1$, $Q_2 = C_1$, $\tilde{\lambda} = \lambda_1$, we obtain

$$e = (\Gamma\tau)^2 = (\Gamma\lambda_1)(\tau\Gamma\lambda_1)^{-1} = (\Gamma\lambda_1)^{-n_1/2},$$

which is the desired result.

3. The group $\text{Hom}(A, J)B^1(A, G)$. Let $\text{Hom}(A, J)$ be the group of all homomorphisms of A into J , the center of H . Since J is the group of characteristic elements of G it follows that $\text{Hom}(A, J)$ is a subgroup of $Z^1(A, G)$. In this section and the next we determine the factor group $Z^1(A, G)/\text{Hom}(A, J) \cdot B^1(A, G)$. We need the following characterization of the group $\text{Hom}(A, J) \cdot B^1(A, G)$.

THEOREM 1. *Let $\Gamma \in Z^1(A, G)$. Then $\Gamma \in \text{Hom}(A, J)B^1(A, G)$ if and only if the following conditions hold:*

- (i) $\Gamma\lambda_1 \in C_1$.
- (ii) $\Gamma\theta_1 \in C_1 \times C_2$.
- (iii) $\Gamma\eta_2 \in J \times C_2 \times C_3$.
- (iv) *Either $n_1 \neq 4n_2$ or $\Gamma(\lambda_1\eta_1) \in C \times C_3$, where C is the cyclic group generated by $c_1^2c_2^{-1}$.*

Proof. Throughout this proof i and j will always denote positive integers. We have $J \subseteq C_1$. Furthermore if $n_1 = 4n_2$, then $J = C_1 \cap C \subseteq C$. It follows that (i), (ii), (iii), and (iv) hold for any $\Gamma' \in \text{Hom}(A, J)$. Let $\Gamma'' \in B^1(A, G)$. Then there is a fixed $a \in G$ such that $\Gamma''\sigma = a\sigma a^{-1}$ for all $\sigma \in A$. Now $g\lambda_1g^{-1} \in C_1$, $g\theta_1g^{-1} \in C_1$, $g\eta_2g^{-1} \in C_3$, and $g\lambda_1\eta_1g^{-1} \in C$ for all $g \in G$. Hence (i), (ii), (iii), and (iv) hold for any $\Gamma'' \in B^1(A, G)$. Now the conditions (i), (ii), (iii), and (iv) are of such a nature that if they hold for two cocycles Γ' and Γ'' , then they hold for their product $\Gamma'\Gamma''$. It follows that (i), (ii), (iii), and (iv) hold for every element of $\text{Hom}(A, J)B^1(A, G)$.

To prove the converse let $\Gamma \in Z^1(A, G)$ and suppose that (i), (ii), (iii), (iv) hold. According to Lemma 6 we can write, for $1 \leq i < k$,

$$\Gamma\theta_i = c_i^{u_i} c_{i+1}^{v_i} h_i,$$

where $h_i \in K_{i,i+1}$ and u_i, v_i are integers. We have $h_1 = e$ by (ii). Now $\Gamma\lambda_i$

$\in C_1 \times C_j$ for $j \geq 2$ by Lemmas 6 and 5. Furthermore $\Gamma\lambda_1 \in C_1$ by (i). Therefore $\theta_i \Gamma\lambda_i = \Gamma\lambda_i$ for all i . We observe that $\lambda_i = \theta_i \lambda_i \theta_i$. Hence

$$\Gamma\lambda_i = (\Gamma\theta_i)(\theta_i \Gamma\lambda_i)(\theta_i \lambda_i \Gamma\theta)$$

which reduces to

$$e = (\Gamma\theta_i)(\theta_i \lambda_i \Gamma\theta_i) = c_i^{r_i v_i} c_{i+1}^{2v_i}.$$

Thus $n_i | r_i v_i$ which is equivalent to $n_{i+1} | v_i$. Therefore $c_{i+1}^{v_i} = e$ and we have

$$(7) \quad \Gamma\theta_i = c_i^{u_i} h_i.$$

It follows at once that $\theta_i \Gamma\theta_i = \Gamma\theta_i$. Combining this with $\theta_i^{n_i/r_i} = e$ we obtain

$$e = \Gamma(\theta_i^{n_i/r_i}) = (\Gamma\theta_i)^{n_i/r_i}.$$

Therefore $c_i^{n_i u_i/r_i} = e$ and we have $r_i | u_i$.

By Lemmas 6 and 5 we have $\Gamma\lambda' \in J^{(2)} \times G' = J \times G'$. Every element of G' has odd order. Therefore every element of G' is a square and we have

$$\Gamma\lambda' = h g'^2,$$

where $h \in J$ and $g' \in G'$.

By Lemmas 6 and 5 we can write

$$\Gamma\eta_1 = c_1^r c_2^s \bar{h},$$

where $\bar{h} \in K_{1,2} \subseteq C_3$. By (i) we have $\Gamma\lambda_1 = c_1^t$ for some integer t . By Lemma 8, t is even if $n_1 \geq 8n_2$. Now put

$$s' = \begin{cases} s & \text{if } n_1 \leq 4n_2, \\ -t/2 & \text{if } n_1 \geq 8n_2, \end{cases}$$

and

$$f = g'^{-1} c_1^{s'} \prod_{i=1}^{k-1} c_{i+1}^{u_i/r_i}.$$

Let Γ_f be the element of $B^1(A, G)$ corresponding to f , i.e. $\Gamma_f \sigma = f \sigma f^{-1}$ for all $\sigma \in A$. Put $\Gamma_1 = \Gamma \Gamma_f$. To prove Theorem 1 it is sufficient to show that $\Gamma_1 \in \text{Hom}(A, J)$. As a result of the choice of f we have

$$(8) \quad \Gamma_1 \lambda' = h \in J,$$

$$(9) \quad \Gamma_1 \theta_1 = h_1 = e,$$

and

$$(10) \quad \Gamma_1 \theta_j = h_j \in K_{j,j+1} \subseteq C_1 \quad \text{for all } j \geq 2.$$

Furthermore

$$(11) \quad \Gamma_1 \eta_1 = c_1 \bar{h} \text{ if } n_1 \leq 4n_2$$

and

$$(12) \quad \Gamma_1 \lambda_1 = e \text{ if } n_1 \geq 8n_2.$$

We note also that (i), (ii), (iii), and (iv) hold for Γ_1 since they hold for both Γ and Γ_f .

Let $\tau' \in A'$. Applying Lemma 7 to the case $Q_2 = G'$, $Q_1 = G^{(2)}$, $\bar{\lambda} = \lambda'$ we obtain

$$(13) \quad (\Gamma_1 \tau')^2 = (\Gamma_1 \lambda')(\tau' \Gamma_1 \lambda')^{-1} = h(\tau' h)^{-1} = e.$$

Now $\Gamma_1 \tau' \in J^{(2)} \times G' = J \times G'$ by Lemmas 6 and 5. Combining this with (13) we see that $\Gamma_1 \tau' \in J$ for all $\tau' \in A'$.

Since $A = A' \times A^{(2)}$ and since the automorphisms λ_i , ξ_i , θ_i , η_i generate $A^{(2)}$, it is sufficient to show that $\Gamma_1 \lambda_i$, $\Gamma_1 \xi_i$, $\Gamma_1 \theta_j$, and $\Gamma_1 \eta_j$ are elements of J for $1 \leq i \leq k$, $1 \leq j < k$.

Now let u be an odd integer and let τ_i be the automorphism of C_i such that $\tau_i c_i = c_i^u$. Since $h_i^2 = e$ and $\theta_i h_i = h_i$ we have $\Gamma_1(\theta_i^u) = (\Gamma_1 \theta_i)^u = h_i^u = h_i$. It follows that $\tau_{i+1} \Gamma_1(\theta_i^u) = \Gamma_1 \theta_i$. Hence if we apply Γ_1 to both sides of the identity $\theta_i \tau_{i+1} = \tau_{i+1} \theta_i^u$, we obtain $\theta_i \Gamma_1 \tau_{i+1} = \Gamma_1 \tau_{i+1}$. Now $\Gamma_1 \tau_{i+1} \in K_{i+1} \times C_{i+1}$, $K_{i+1} \subseteq C_1$, and e is the only element of C_{i+1} left fixed by θ_i . Therefore $\Gamma_1 \tau_{i+1} \in K_{i+1}$. In particular

$$(14) \quad \Gamma_1 \lambda_j \in K_j \quad \text{and} \quad \Gamma_1 \xi_j \in K_j \quad \text{for all } j \geq 2.$$

We have $\Gamma_1 \theta_1 = \Gamma_1(\theta_1^5) = e$. Therefore if we apply Γ_1 to both sides of $\theta_1^5 \xi_1 = \xi_1 \theta_1$, we obtain

$$\theta_1^5 \Gamma_1 \xi_1 = \Gamma_1 \xi_1.$$

Since $\Gamma_1 \xi_1 \in C_1 \times K_1 \subseteq C_1 \times C_2$ this implies that

$$(15) \quad \Gamma_1 \xi_1 \in C_1.$$

Now we study $\Gamma_1 \eta_i$. By Lemma 6 we have

$$\Gamma_1 \eta_i \in C_i \times C_{i+1} \times K_{i,i+1}.$$

Applying Γ_1 to both sides of $\eta_{i+1} \theta_i = \theta_i \eta_{i+1}$ and noting that $\eta_{i+1} \Gamma_1 \theta_i = \Gamma_1 \theta_i$, we obtain $\Gamma_1 \eta_{i+1} = \theta_i \Gamma_1 \eta_{i+1}$. It follows that

$$(16) \quad \Gamma_1 \eta_j \in C_{j+1} \times K_{j,j+1} \subseteq C_1 \times C_{j+1} \quad \text{for all } j \geq 2.$$

Now put $\psi_{i+1} = (\theta_{i+1} \lambda_{i+1} \theta_i)^2$. By (9), (10) and (14), $\Gamma_1(\theta_{i+1} \lambda_{i+1} \theta_i)$ is an element of C_1 of order at most 2. Hence $\Gamma_1 \psi_{i+1} = e$. Now

$$\psi_{i+1} c_j = \begin{cases} c_j & \text{if } j \neq i+2, \\ c_i^{r_i r_{i+1}} c_{i+2} & \text{if } j = i+2. \end{cases}$$

It follows that $\psi_{i+1}g = g$ if and only if $g \in N_{i+2}$. Furthermore $g^{-1}\psi_{i+1}g \in C_i$ for all $g \in G$. Now

$$\psi_{i+1}\eta_{i+1} = \eta_{i+1}\theta_i^{r_{i+1}}\psi_{i+1}.$$

We apply Γ_1 to both sides and get

$$\psi_{i+1}\Gamma_1\eta_{i+1} = (\Gamma_1\eta_{i+1})\eta_{i+1}\Gamma_1(\theta_i^{r_{i+1}}) = (\Gamma_1\eta_{i+1})h_i^{r_{i+1}}.$$

Hence $h_i^{r_{i+1}}$ is of the form $g^{-1}\psi_{i+1}g$. Therefore $h_i^{r_{i+1}} \in C_i$. Since $h_1 = e$ and $h_j \in C_1$ for $j \geq 2$, we have $h_i^{r_{i+1}} = e$ for all i . Therefore $\Gamma_1\eta_{i+1} \in N_{i+2}$. Combining this with (16) we have

$$\Gamma_1\eta_j \in K_{j,j+1} \subseteq C_1 \quad \text{for all } j \geq 2.$$

By Lemma 5 we have $K_{j,j+1} = J$ if $j \geq 3$. Furthermore $\Gamma_1\eta_2 \in J \times C_2 \times C_3$ by (iii). Therefore

$$(17) \quad \Gamma_1\eta_j \in J \quad \text{for all } j \geq 2.$$

If we apply Γ_1 to both sides of the identities $\lambda_2 = \eta_1\lambda_2\eta_1$ and $\theta_2\eta_1 = \eta_1\theta_2$ we obtain

$$(18) \quad (\Gamma_1\lambda_2)(\eta_1\Gamma_1\lambda_2)^{-1} = (\Gamma_1\eta_1)(\eta_1\lambda_2\Gamma_1\eta_1)$$

and

$$(19) \quad (\Gamma_1\theta_2)(\eta_1\Gamma_1\theta_2)^{-1} = (\Gamma_1\eta_1)(\theta_2\Gamma_1\eta_1)^{-1}.$$

Next we will establish:

$$(20) \quad \Gamma_1\lambda_2 \in J, \quad \Gamma_1\theta_2 \in J$$

and

$$(21) \quad \Gamma_1\eta_1 \in J \times C_2.$$

There are two cases to be considered.

(I) $n_1 > n_2$. Here $K_{2,3} = K_2 = J$ by Lemma 5. Hence (20) follows from (14) and (10). We have $\Gamma_1\eta_1 = c_1^r c_2^{s'} \bar{h}$, where r and s' are integers and $\bar{h} \in K_{1,2} \subseteq C_3$, $\bar{h}^2 = e$. It follows from (20) that the left hand sides of both (18) and (19) are equal to e . Therefore

$$e = (\Gamma_1\eta_1)(\eta_1\lambda_2\Gamma_1\eta_1) = c_1^{2r} c_2^r$$

and

$$e = (\Gamma_1\eta_1)(\theta_2\Gamma_1\eta_1)^{-1} = \bar{h}(\theta_2\bar{h})^{-1}.$$

Therefore $c_1^{2r} = \bar{h} = e$ which implies (21).

(II) $n_1 = n_2$. In this case let ζ be the automorphism of $C_1 \times C_2$ such that $\zeta c_1 = c_2$, $\zeta c_2 = c_1$. We have $\zeta^2 = e$ and $\eta_1 = \zeta\theta_1\zeta$. Applying Γ_1 to these two equalities we obtain $\Gamma_1\zeta = \zeta(\Gamma_1\zeta)^{-1}$ and

$$\Gamma_1\eta_1 = (\Gamma_1\zeta)(\zeta\Gamma_1\theta_1)(\zeta\theta_1\Gamma_1\zeta) = \zeta\{(\Gamma_1\zeta)^{-1}(\theta_1\Gamma_1\zeta)\},$$

since $\Gamma_1\theta_1 = e$ by (9). Now $g^{-1}\theta_1g \in C_1$ and hence $\zeta(g^{-1}\theta_1g) \in C_2$ for all $g \in G$. Therefore $\Gamma_1\eta_1 \in C_2$ and (21) is established. It follows from (21) that the right hand sides of both (18) and (19) are equal to e . Hence $\Gamma_1\lambda_2 = \eta_1\Gamma_1\lambda_2$ and $\Gamma_1\theta_2 = \eta_1\Gamma_1\theta_2$. Since we already have $\Gamma_1\lambda_2 \in C_1$ by (14) and $\Gamma_1\theta_2 \in C_1$ by (10) these last two equalities imply (20).

We have established that (20) and (21) hold in all cases.

We will next establish:

$$(22) \quad \Gamma_1\lambda_1 \in J,$$

$$(23) \quad \text{either } n_1 = n_2 \text{ or } \Gamma_1\xi_1 \in J,$$

and

$$(24) \quad \Gamma_1\eta_1 \in J.$$

If $r_1 \geq 8$ then (22) follows from (12). If $r_1 \leq 4$ then (24) follows from (11) and (21). We will now show that (22), (23) and (24) are equivalent. Then it will follow that all three hold in all cases.

Suppose (22) holds. We apply Lemma 7 with $Q_2 = C_1$, $Q_1 = N_1$, and $\bar{\lambda} = \lambda_1$. Then (5) yields $(\Gamma_1\xi_1)^2 = e$. Combining this with (15) we see that $\Gamma_1\xi_1 \in J$ if $n_1 > n_2$. Thus (22) implies (23).

Next suppose (23) holds and that $r_1 \geq 8$. We specialize τ_i to be the automorphism of C_i such that $\tau_i c_i = c_i^{1+r_1}$. Since $8 \mid r_1$ it follows that τ_i is an even power of ξ_i . Furthermore, since $n_1 > n_2$, it follows from (14) and (23) that $\Gamma_1\xi_i \in J$ for all i . Therefore $\Gamma_1\tau_i = e$ for all i . Now

$$(\eta_1\lambda_2\tau_2\theta_1)^2 = \tau_1\tau_2.$$

We apply Γ_1 to both sides of this identity, noting that $\Gamma_1\lambda_2 \in K_2 = J$, $\Gamma_1\theta_1 = \Gamma_1\tau_1 = \Gamma_1\tau_2 = e$, and $\Gamma_1\eta_1 = c_1'c_2''$, where $c_1' \in J$ by (21). This gives us

$$e = (\Gamma_1\eta_1)(\eta_1\lambda_2\tau_2\theta_1\Gamma_1\eta_1) = c_1'^{r_1s''}.$$

Therefore $n_1 \mid r_1s''$, $n_2 \mid s''$, $c_2^{s''} = e$, and $\Gamma_1\eta_1 = c_1' \in J$. Thus (23) implies (24).

Suppose (24) holds. Then applying Γ_1 to $(\lambda_1\eta_1)^2 = e$, we obtain

$$e = (\Gamma_1\lambda_1)(\lambda_1\eta_1\Gamma_1\lambda_1) = c_2'',$$

where $\Gamma_1\lambda_1 = c_1'$. If $r_1 \leq 2$, then $c_2'' = e$ implies $c_1' \in J$. If $r_1 = 4$ then $\Gamma_1(\lambda_1\eta_1) \in C \times C_3$ by (iv), and, since $\lambda_1\Gamma_1\eta_1 \in J \subseteq C$, we have $\Gamma_1\lambda_1 \in (C \times C_3) \cap C_1 = J$. We already know that (22) holds if $r_1 \geq 8$. Therefore (24) implies (22).

It follows that (22), (23), and (24) hold in all cases.

Combining (17) and (24) we have

$$\Gamma_1\eta_j \in J, \quad 1 \leq j < k.$$

From (9), (10), and (20) we obtain

$$\Gamma_1 \theta_j \in J, \quad 1 \leq j < k.$$

From (14), (20), and (22) we obtain

$$\Gamma_1 \lambda_i \in J, \quad 1 \leq i \leq k,$$

and

$$\Gamma_1 \xi_i \in J, \quad 3 \leq i \leq k.$$

Furthermore (14) and (23) imply: If $n_1 > n_2$, then $\Gamma_1 \xi_1 \in J$ and $\Gamma_1 \xi_2 \in J$. To complete the proof of Theorem 1 we need only show that $\Gamma_1 \xi_1 \in J$ and $\Gamma_1 \xi_2 \in J$ for the case $n_1 = n_2$.

Suppose $n_1 = n_2$. In this case J consists of e alone. As above let ζ be the automorphism of $C_1 \times C_2$ such that $\zeta c_1 = c_2$ and $\zeta c_2 = c_1$. Since $\zeta = \lambda_1 \eta_1 \theta_1^{-1} \eta_1$ it follows that $\Gamma_1 \zeta = e$. Hence, if we apply Γ_1 to both sides of the identity $\xi_2 = \zeta \xi_1 \zeta$ we obtain $\Gamma_1 \xi_2 = \zeta \Gamma_1 \xi_1$. Now, by (14) and (15), $\Gamma_1 \xi_2 \in C_1$ and $\zeta \Gamma_1 \xi_1 \in C_2$. Hence $\Gamma_1 \xi_1 = \Gamma_1 \xi_2 = e \in J$ and the proof of Theorem 1 is complete.

Let Z_i be the set of all cocycles that satisfy condition (i) of Theorem 1. Clearly Z_i is a subgroup of $Z^1(A, G)$ and we put $m_i = [Z^1(A, G) : Z_i]$, the index of Z_i in $Z^1(A, G)$.

LEMMA i. $m_i \leq 2$. If $m_i = 2$, then either $n_1 > n_2 > n_3$ or $n_1 = n_2 \geq 4n_3$.

Proof. The inequality $m_i \leq 2$ follows at once from Lemma 6. Suppose $m_i = 2$ and let $\Gamma \in Z^1(A, G)$, $\Gamma \notin Z_i$. Then $\Gamma \lambda_1 \notin C_1$. From Lemma 6 we have $\Gamma \lambda_1 \in C_1 \times K_1$. Hence K_1 is not trivial and Lemma 5 yields $n_2 > n_3$. Thus $n_1 \geq n_2 \geq 2n_3$. If $n_1 > 2n_3$ then either $n_1 > n_2 > n_3$ or $n_1 = n_2 \geq 4n_3$. Thus we need only eliminate the case $n_1 = 2n_3$.

Suppose $n_1 = 2n_3$. Then $\lambda_1 \lambda_3 = \phi^2$, where ϕ is the automorphism of $C_1 \times C_3$ such that $\phi c_1 = c_1 c_3$ and $\phi c_3 = c_1^{-2} c_3^{-1}$. It follows from Lemma 6 that

$$(\Gamma \lambda_1)(\lambda_1 \Gamma \lambda_3) = \Gamma(\lambda_1 \lambda_3) = \Gamma(\phi^2) \in C_1 \times C_3.$$

Now $\Gamma \lambda_3 \in K_3 \times C_3 \subseteq C_1 \times C_3$. Hence $\lambda_1 \Gamma \lambda_3 \in C_1 \times C_3$ and we have $\Gamma \lambda_1 \in C_1 \times C_3$. Now $\Gamma \lambda_1 \in C_1 \times K_1 \subseteq C_1 \times C_2$ by Lemmas 6 and 5. Therefore $\Gamma \lambda_1 \in C_1$, a contradiction. This contradiction establishes Lemma i.

Let Z_{ii} be the group of all cocycles Γ satisfying the conditions (i) and (ii) of Theorem 1. Put $m_{ii} = [Z_i : Z_{ii}]$.

LEMMA ii. $m_{ii} \leq 2$. If $m_{ii} = 2$ then $n_3 > n_4$.

Proof. By Lemma 6 the set of all cocycles that satisfy (ii) form a subgroup Z'' of $Z^1(A, G)$ of index at most two. We have $Z_{ii} = Z_i \cap Z''$ and hence

$$m_{ii} = [Z_i : Z_i \cap Z''] = [Z_i Z'' : Z''] \leq [Z^1(A, G) : Z''] \leq 2.$$

If $n_3 = n_4$, then (ii) is satisfied by all $\Gamma \in Z^1(A, G)$ by Lemmas 6 and 5. Hence if $m_{ii} = 2$ then $n_3 > n_4$.

Let Z_{iii} be the group of all cocycles Γ satisfying conditions (i), (ii), and (iii) of Theorem 1. Put $m_{iii} = [Z_{ii}:Z_{iii}]$.

LEMMA iii. $m_{iii} \leq 2$. If $m_{iii} = 2$ then $n_1 = n_2 > n_3 > n_4$.

Proof. By Lemma 6, $\Gamma\eta_2 \in C_2 \times C_3 \times K_{2,3}$ for all $\Gamma \in Z^1(A, G)$. If $n_1 > n_2$, then $K_{2,3} = J$, (iii) holds for all $\Gamma \in Z^1(A, G)$, and $m_{iii} = 1$. Thus without loss of generality we suppose $n_1 = n_2$. Then J is trivial. It follows from Lemma 6 that the set of all cocycles that satisfy (iii) form a subgroup Z''' of index at most two in $Z^1(A, G)$. We have

$$m_{iii} = [Z_{ii}:Z_{ii} \cap Z'''] = [Z_{ii}Z''':Z'''] \leq [Z^1(A, G):Z'''] \leq 2.$$

Now suppose $n_3 = n_4$. Then $\eta_2 = \phi^2$ where ϕ is the automorphism of $C_2 \times C_3 \times C_4$ such that $\phi c_2 = c_2 c_4$, $\phi c_3 = c_3$, $\phi c_4 = c_3 c_4^{-1}$. It now follows from Lemma 6 that (iii) is satisfied by all $\Gamma \in Z^1(A, G)$, and $m_{iii} = 1$. Thus $n_3 = n_4$ implies $m_{iii} = 1$.

Suppose $n_1 = n_2 = n_3$. Let $\Gamma \in Z_{ii}$ and let ζ be the automorphism of $C_1 \times C_3$ such that $\zeta c_1 = c_3$ and $\zeta c_3 = c_1$. Then $\zeta^2 = \epsilon$ and $\eta_2 = \zeta \theta_1 \zeta$. Applying Γ to these two equalities we obtain $\Gamma \zeta = (\zeta \Gamma \zeta)^{-1}$ and

$$\Gamma \eta_2 = (\Gamma \zeta)(\zeta \Gamma \theta_1)(\zeta \theta_1 \Gamma \zeta) = \zeta \{ (\Gamma \theta_1)(\Gamma \zeta)^{-1}(\theta_1 \Gamma \zeta) \}.$$

Now $\Gamma \theta_1 \in C_1 \times C_2$ by (ii). Furthermore $g^{-1} \theta_1 g \in C_1$ for all $g \in G$. Therefore

$$(\Gamma \theta_1)(\Gamma \zeta)^{-1}(\theta_1 \Gamma \zeta) \in C_1 \times C_2$$

and $\Gamma \eta_2 \in C_2 \times C_3$. Thus $m_{iii} = 1$ if $n_1 = n_2 = n_3$.

It follows that if $m_{iii} > 1$ then $n_1 = n_2 > n_3 > n_4$, and Lemma iii is established.

Now put $m_{iv} = [Z_{iii}:\text{Hom}(A, J)B^1(A, G)]$.

LEMMA iv. $m_{iv} \leq 2$. If $m_{iv} = 2$ then $n_1 = 4n_2$.

Proof. If $n_1 \neq 4n_2$, then $m_{iv} = 1$ by Theorem 1, and Lemma iv holds. Suppose $n_1 = 4n_2$ and let $\Gamma \in Z_{iii}$. By Theorem 1, $\Gamma \in \text{Hom}(A, J)B^1(A, G)$ if and only if $\Gamma(\lambda_1 \eta_1) \in C \times C_3$. Since $\Gamma(\lambda_1 \eta_1) \in C_1 \times C_2 \times K_{1,2}$ we have $\Gamma(\lambda_1 \eta_1) = c_1^r (c_1^2 c_2^{-1})^s \bar{h}$, where r and s are integers and $\bar{h}$ is an element of C_3 such that $\bar{h}^2 = e$. Now $(\lambda_1 \eta_1)^2 = \epsilon$. Hence

$$e = \Gamma(\lambda_1 \eta_1) \lambda_1 \eta_1 \Gamma(\lambda_1 \eta_1) = c_2^r,$$

and we have $n_2 \mid r$. Now in the case under discussion $C \cap C_1 = J$ and J is the cyclic group generated by $c_1^{2n_2}$. Let D be the group generated by $c_1^{2n_2}$ and $c_1^2 c_2^{-1}$. Then $\Gamma \rightarrow c_1^r C$ is a homomorphism of Z_{iii} into D/C with kernel $\text{Hom}(A, J)B^1(A, G)$. Hence $m_{iv} \leq [D:C] = 2$, which establishes Lemma iv.

Clearly $[Z^1(A, G):\text{Hom}(A, J)B^1(A, G)] = m_i m_{iii} m_{iii} m_{iv}$. It follows from Lemmas iii and iv that m_{iii} and m_{iv} cannot both be two. Therefore

$$[Z^1(A, G):\text{Hom}(A, J)B^1(A, G)] \leq 8.$$

It will follow from the results of §4 that this index can actually assume any of the four possible values 1, 2, 4, 8.

4. Special cocycles. The four lemmas of §3 give us necessary conditions for $m_v = 2$, $v = i, ii, iii$, or iv . In this section we will show, by actual construction of suitable cocycles, that these conditions are also sufficient. This will provide us with a set of generators of the factor group

$$Z^1(A, G)/\text{Hom}(A, J)B^1(A, G).$$

Let $\bar{G}$ be a characteristic subgroup of G , and let $\bar{A}$ be the group of automorphisms of $\bar{G}$. For any $\sigma \in A$ we let $\bar{\sigma}$ denote the restriction of σ to $\bar{G}$. Since $\bar{G}$ is a characteristic subgroup of G it follows that $\bar{\sigma} \in \bar{A}$. In particular $\bar{e}$ is the identity of $\bar{A}$. If $\bar{\Gamma} \in Z^1(\bar{A}, \bar{G})$, then there is a corresponding element $\Gamma \in Z^1(A, G)$ such that

$$(25) \quad \Gamma\sigma = \bar{\Gamma}\bar{\sigma} \quad \text{for all } \sigma \in A.$$

Let $\alpha_1, \alpha_2, \dots, \alpha_m$ be a set of generators of $\bar{A}$ and let $\tilde{\Gamma}$ be a mapping of this set of generators into $\bar{G}$. We will say that $\tilde{\Gamma}$ satisfies the relation⁽²⁾

$$\alpha_{l_1}\alpha_{l_2} \cdots \alpha_{l_r} = \bar{e}$$

if

$$(\tilde{\Gamma}\alpha_{l_1})(\alpha_{l_1}\tilde{\Gamma}\alpha_{l_2}) \cdots (\alpha_{l_1}\alpha_{l_2} \cdots \alpha_{l_{r-1}}\tilde{\Gamma}\alpha_{l_r}) = e.$$

Let

$$(26) \quad \begin{cases} \alpha_{l_1} \cdots \alpha_{l_r} = \bar{e}, \\ \vdots \\ \alpha_{q_1} \cdots \alpha_{q_u} = \bar{e} \end{cases}$$

be a set of relations between $\alpha_1, \dots, \alpha_m$ that is complete in the sense that any other relation can be deduced from it. The following lemma is due to Anne P. Cobbe [1, pp. 43-45].

LEMMA 9. *If $\tilde{\Gamma}$ satisfies each of the relations (26), then $\tilde{\Gamma}$ can be extended to an element $\bar{\Gamma} \in Z^1(\bar{A}, \bar{G})$. The extension $\bar{\Gamma}$ is uniquely determined by $\tilde{\Gamma}$.*

Throughout the remainder of the paper we will use the following additional notation:

$$a_i = c_i^{n_i/4}$$

for those i for which $n_i \geq 4$, and

$$b_j = c_j^{n_j/2}, \quad 1 \leq j \leq k.$$

We are now in a position to proceed to the construction of special cocycles.

⁽²⁾ For our present purposes it is sufficient to consider only relations in which no negative powers of the α_i occur.

(i) By Lemma i, if $m_i=2$, then either $n_1>n_2>n_3$ or $n_1=n_2\geq 4n_3$.

(i') Suppose $n_1>n_2>n_3$. Here we take $\bar{G}$ to be the characteristic subgroup of G generated by a_1 and b_2 . Then $\bar{G}$ is the direct product of a cyclic group of order 4 and one of order 2. Hence $\bar{A}$ is the octic group. Let α and β be the automorphisms of $\bar{G}$ defined by

$$\begin{aligned}\alpha a_1 &= a_1 b_2, & \alpha b_2 &= a_1^2 b_2, \\ \beta a_1 &= a_1^3 b_2, & \beta b_2 &= b_2.\end{aligned}$$

Then $\alpha^2=\bar{\lambda}_1$ and $\alpha\beta=\bar{\theta}_1$, the restrictions of λ_1 and θ_1 respectively to $\bar{G}$. Furthermore α and β generate $\bar{A}$ and a complete set of relations is

$$(27) \quad \alpha^4 = \beta^2 = (\alpha\beta)^2 = \bar{\epsilon}.$$

The mapping $\bar{\Gamma}'$ of the set $\{\alpha, \beta\}$ into $\bar{G}$ given by

$$\bar{\Gamma}'\alpha = a_1 b_2, \quad \bar{\Gamma}'\beta = e$$

satisfies (27). Hence, by Lemma 9, $\bar{\Gamma}'$ can be extended to an element $\bar{\Gamma}'$ of $Z^1(\bar{A}, \bar{G})$. This in turn corresponds to an element $\Gamma' \in Z^1(A, G)$ by the correspondence (25). Now

$$\Gamma'\lambda_1 = \bar{\Gamma}'(\alpha^2) = (\bar{\Gamma}'\alpha)(\alpha\bar{\Gamma}'\alpha) = b_2 \notin C_1.$$

Thus $\Gamma' \notin Z_i$.

(i''). Suppose $n_1=n_2\geq 4n_3$. Here we take $\bar{G}$ to be the characteristic subgroup of G generated by a_1 and a_2 . In this case $\bar{G}$ is the direct product of two cyclic groups of order four. Hence $\bar{A}$ has order 96. Let γ and δ be the automorphisms of $\bar{G}$ such that

$$\begin{aligned}\gamma a_1 &= a_1^2 a_2, & \gamma a_2 &= a_1 a_2^2, \\ \delta a_1 &= a_2^3, & \text{and } \delta a_2 &= a_1 a_2.\end{aligned}$$

Then γ has order 2, δ has order 6, and $\gamma\delta^2$ has order 4. The symmetric group of order 24 is characterized, as an abstract group, by the property that it is generated by an element of order 2 and an element of order 3 whose product has order 4. Hence γ and δ^2 generate a group $\bar{A}_{24}$ of order 24 isomorphic to the symmetric group. Now $(\gamma\delta^3)^2 = \bar{\epsilon}$. It follows that δ^3 commutes with every element of $\bar{A}_{24}$. Therefore $\delta^3 \in \bar{A}_{24}$. It follows that γ and δ generate a group $\bar{A}_{48}$ of order 48 and that a complete set of relations (3) for this group is

$$(28) \quad \gamma^2 = \delta^6 = (\gamma\delta^2)^4 = (\gamma\delta^3)^2 = \bar{\epsilon}.$$

Now

$$(29) \quad (\bar{\lambda}_1 \gamma)^2 \delta^3 = \bar{\epsilon}.$$

(*) The group $\bar{A}_{48}$ is isomorphic to the direct product of the symmetric group of order 24 and the group of order 2. The set of relations (28) for this group is due to G. A. Miller [5].

It follows that $(\bar{\lambda}_1\gamma)^2 \notin \bar{A}_{24}$, and hence $\bar{\lambda}_1 \notin \bar{A}_{48}$. Therefore γ , δ , and $\bar{\lambda}_1$ generate $\bar{A}$. A complete (though redundant) set of relations for $\bar{A}$, with respect to this set of generators, is (28), (29), and

$$\bar{\lambda}_1^2 = \bar{\lambda}_1\delta\bar{\lambda}_1\gamma\delta\gamma = \bar{\epsilon}.$$

This set of relations is satisfied by the mapping $\tilde{\Gamma}''$ of $\{\gamma, \delta, \bar{\lambda}_1\}$ into $\bar{G}$ given by

$$\tilde{\Gamma}''\gamma = \tilde{\Gamma}''\bar{\lambda}_1 = a_1^2 a_2^2, \quad \tilde{\Gamma}''\delta = e.$$

By Lemma 9, $\tilde{\Gamma}''$ can be extended to an element $\tilde{\Gamma}'' \in Z^1(\bar{A}, \bar{G})$, and $\tilde{\Gamma}''$ in turn corresponds to a cocycle $\Gamma'' \in Z^1(A, G)$. Clearly

$$\Gamma''\lambda_1 = \tilde{\Gamma}''\bar{\lambda}_1 = a_1^2 a_2^2 \notin C_1.$$

Hence $\Gamma'' \notin Z_1$.

Combining Lemma i with the existence of Γ' and Γ'' we see that

$$m_i = \begin{cases} 2 & \text{if } n_1 > n_2 > n_3, \\ 2 & \text{if } n_1 = n_2 \geq 4n_3, \\ 1 & \text{otherwise.} \end{cases}$$

Put

$$\Gamma_i = \begin{cases} \Gamma' & \text{if } n_1 > n_2 > n_3, \\ \Gamma'' & \text{if } n_1 = n_2 \geq 4n_3. \end{cases}$$

Then Γ_i is an element of $Z^1(A, G)$, defined whenever $m_i=2$, such that $\Gamma_i \notin Z_1$.

(ii) By Lemma ii, if $m_{ii}=2$, then $n_3 > n_4$. Suppose $n_3 > n_4$. We now take $\bar{G}$ to be the characteristic subgroup of G generated by b_1 , b_2 , and b_3 . This $\bar{G}$ is the direct product of three groups of order two, and the corresponding $\bar{A}$ has order 168. Let ϕ and ρ be the automorphisms of $\bar{G}$ such that

$$\begin{aligned} \phi b_1 &= b_1 b_3, & \phi b_2 &= b_3, & \phi b_3 &= b_1 b_2 b_3, \\ \rho b_1 &= b_1 b_2, & \rho b_2 &= b_2 b_3, & \rho b_3 &= b_2. \end{aligned}$$

We note that $\phi\rho = \bar{\theta}_1$, the restriction of θ_1 to $\bar{G}$. Furthermore $\rho\phi^5 b_1 = b_3$, $\rho\phi^5 b_2 = b_1 b_2$, and $\rho\phi^5 b_3 = b_1$. Hence

$$(30) \quad \phi^7 = \rho^3 = (\phi\rho)^2 = (\rho\phi^5)^4 = \bar{\epsilon}.$$

It is known⁽⁴⁾ that (30) and $\phi \neq \bar{\epsilon}$ are sufficient to insure that ϕ and ρ generate the simple group of order 168. Therefore ϕ and ρ are a set of gener-

⁽⁴⁾ This set of generators and relations for the simple group of order 168 is due to Dyck [2, p. 41] and confirmed by Miller [5].

ators for $\bar{A}$ and (30) is a corresponding complete set of relations. Let $\tilde{\Gamma}_{ii}$ be the mapping of $\{\phi, \rho\}$ into $\bar{G}$ given by

$$\tilde{\Gamma}_{ii}\phi = e, \quad \tilde{\Gamma}_{ii}\rho = b_2.$$

It can be readily verified that $\tilde{\Gamma}_{ii}$ satisfies (30). Hence $\tilde{\Gamma}_{ii}$ can be extended to an element $\bar{\Gamma}_{ii} \in Z^1(\bar{A}, \bar{G})$. By the correspondence (25), $\bar{\Gamma}_{ii}$ corresponds to a cocycle $\Gamma_{ii} \in Z^1(A, G)$. Now $\bar{\lambda}_1 = \bar{e}$. Hence $\Gamma_{ii}\lambda_1 = e$. Furthermore

$$\Gamma_{ii}\theta_1 = \bar{\Gamma}_{ii}(\phi\rho) = \phi\tilde{\Gamma}_{ii}\rho = b_3 \in C_1 \times C_2.$$

Thus we have constructed a cocycle Γ_{ii} , defined whenever $n_3 > n_4$, such that

$$\Gamma_{ii} \in Z_i, \quad \Gamma_{ii} \in Z_{ii}.$$

Combining the existence of Γ_{ii} with Lemma ii we obtain

$$m_{ii} = \begin{cases} 2 & \text{if } n_3 > n_4, \\ 1 & \text{if } n_3 = n_4. \end{cases}$$

(iii) By Lemma iii, if $m_{iii} = 2$, then $n_1 = n_2 > n_3 > n_4$. Suppose $n_1 = n_2 > n_3 > n_4$. In this case again we take $\bar{G}$ to be the group generated by b_1, b_2 , and b_3 . Let $\bar{\Gamma}_{ii}$ be the element of $Z^1(\bar{A}, \bar{G})$ defined above. Let π be the projection of G onto $C_1 \times C_2$ such that

$$\pi g = \begin{cases} g & \text{if } g \in C_1 \times C_2, \\ e & \text{if } g \in N_{1,2}. \end{cases}$$

Let χ be the homomorphism of G onto $\bar{G}$ such that

$$\chi c_i = \begin{cases} b_i & \text{if } i \leq 3, \\ e & \text{if } i > 3, \end{cases}$$

and $\chi g' = e$ if $g' \in G'$. The kernel of χ is generated by G' and the elements $c_i^2, c_j, 1 \leq i \leq 3, j > 3$. Since $n_3 > n_4$ it follows that the kernel of χ is a characteristic subgroup of G . Therefore if $\sigma \in A$, then

$$\bar{\sigma}\chi g = \chi\sigma g$$

gives a single valued mapping $\bar{\sigma}$ of $\bar{G}$ onto itself. Now $\bar{\sigma} \in \bar{A}$, $\sigma \rightarrow \bar{\sigma}$ is a homomorphism of A into $\bar{A}$, and we put

$$(31) \quad \Gamma_{iii}\sigma = \pi\bar{\Gamma}_{ii}\bar{\sigma}.$$

We must now show that Γ_{iii} is a cocycle. Since $n_1 = n_2 > n_3$ it follows that $\pi\chi g = g^{n_1/2}$ for all $g \in G^{(2)}$. Hence for any $g \in G^{(2)}$ we have

$$\sigma\pi\chi g = (\sigma g)^{n_1/2} = \pi\chi\sigma g = \pi\bar{\sigma}\chi g.$$

Therefore $\sigma\pi\bar{g} = \pi\bar{\sigma}\bar{g}$ for all $\bar{g} \in \bar{G}$. Now let σ and τ be arbitrary elements of A . Since $\bar{\Gamma}_{ii}\bar{\tau} \in \bar{G}$, we have $\sigma\pi\bar{\Gamma}_{ii}\bar{\tau} = \pi\bar{\sigma}\bar{\Gamma}_{ii}\bar{\tau}$. Hence

$$\begin{aligned}
(\Gamma_{iii}\sigma)(\sigma\Gamma_{iii}\tau) &= (\pi\bar{\Gamma}_{ii}\bar{\sigma})(\sigma\pi\bar{\Gamma}_{ii}\bar{\tau}) \\
&= (\pi\bar{\Gamma}_{ii}\bar{\sigma})(\pi\bar{\sigma}\bar{\Gamma}_{ii}\bar{\tau}) \\
&= \pi\{(\bar{\Gamma}_{ii}\bar{\sigma})(\bar{\sigma}\bar{\Gamma}_{ii}\bar{\tau})\} \\
&= \pi\bar{\Gamma}_{ii}(\bar{\sigma}\bar{\tau}) \\
&= \Gamma_{iii}(\sigma\tau).
\end{aligned}$$

Thus $\Gamma_{iii} \in Z^1(A, G)$. We note that if

$$\sigma c_i = \prod_{i=1}^k c_i^{q_{ij}}, \quad 1 \leq j \leq k,$$

then $\Gamma_{iii}\sigma$ depends only on the residue classes of the q_{ij} modulo 2, since these residue classes determine $\bar{\sigma}$ completely. Hence $\Gamma_{iii}\lambda_1 = e$. Furthermore $\Gamma_{iii}\sigma \in C_1 \times C_2$ for all $\sigma \in A$. In particular $\Gamma_{iii}\theta_1 \in C_1 \times C_2$. Finally $\bar{\eta}_2 = \phi^2 \rho \phi^6 \rho \phi^5$, $\bar{\Gamma}_{ii}\bar{\eta}_2 = b_1 b_3$, and hence

$$\Gamma_{iii}\eta_2 = b_1 \notin C_2 \times C_3 = J \times C_2 \times C_3.$$

Thus we have a cocycle Γ_{iii} , defined whenever $n_1 = n_2 > n_3 > n_4$, such that

$$\Gamma_{iii} \in Z_{ii}, \quad \Gamma_{iii} \notin Z_{iii}.$$

It follows that

$$m_{iii} = \begin{cases} 2 & \text{if } n_1 = n_2 > n_3 > n_4, \\ 1 & \text{otherwise.} \end{cases}$$

(iv) Suppose $n_1 = 4n_2$. We now take $\bar{G}$ to be the characteristic subgroup of G of order 4 generated by a_1 . In this case $\bar{A}$ has order two and is generated by $\bar{\lambda}_1$. A complete set of relations is

$$\bar{\lambda}_1^2 = \bar{\epsilon}.$$

It follows from Lemma 9 that there is a cocycle $\bar{\Gamma}_{iv} \in Z^1(\bar{A}, \bar{G})$ such that $\bar{\Gamma}_{iv}\bar{\lambda}_1 = a_1$. By (25), $\bar{\Gamma}_{iv}$ corresponds to a cocycle $\Gamma_{iv} \in Z^1(A, G)$ such that

$$\Gamma_{iv}\sigma = \begin{cases} e & \text{if } \sigma a_1 = a_1, \\ a_1^{-1} & \text{if } \sigma a_1 = a_1^{-1}. \end{cases}$$

We have $\Gamma_{iv}\lambda_1 = a_1$, $\Gamma_{iv}\theta_1 = \Gamma_{iv}\eta_2 = e$, and $\Gamma_{iv}(\lambda_1\eta_1) = a_1 \notin C \times C_3$. Thus we have a cocycle Γ_{iv} , defined when $n_1 = 4n_2$, such that

$$\Gamma_{iv} \in Z_{iii}, \quad \Gamma_{iv} \notin \text{Hom}(A, J)B^1(A, G).$$

It follows that

$$m_{iv} = \begin{cases} 2 & \text{if } n_1 = 4n_2, \\ 1 & \text{if } n_1 \neq 4n_2. \end{cases}$$

We have determined all m_v , $i \leq v \leq iv$, and constructed a cocycle Γ_v for each v such that $m_v = 2$. $Z^1(A, G)$ can be obtained from $\text{Hom}(A, J)B^1(A, G)$ by adjoining these Γ_v ; $H^1(A, G)$ can be obtained from $\text{Hom}(A, J)B^1(A, G)/B^1(A, G)$ by adjoining the corresponding cosets $\Gamma_v B^1(A, G)$.

5. $\text{Hom}(A, J)$. Our next goal is the determination of the group $\text{Hom}(A, J)$. We will discuss the slightly more general problem of the determination of $\text{Hom}(A, T)$, where T is an arbitrary group of order two.

We let e denote the identity of T as well as that of G . Let h be the other element of T . We write

$$G = \prod_p G^{(p)},$$

where p runs over all prime numbers dividing the order of G , and $G^{(p)}$ is the Sylow subgroup of G whose order is a power of p . Now

$$A = \prod_p A^{(p)},$$

where $A^{(p)}$ is the group of automorphisms of $G^{(p)}$. Under suitable identification $\text{Hom}(A^{(p)}, T)$ can be regarded as a subgroup of $\text{Hom}(A, T)$ and we have

$$\text{Hom}(A, T) = \prod_p \text{Hom}(A^{(p)}, T).$$

Now let p be a fixed prime dividing the order of G . We will use the notation of Lemma 1. By Lemma 1, $A^{(p)}$ is generated by the automorphisms γ_j , δ_j , $1 \leq j < l$, and the automorphisms of D_i , $1 \leq i \leq l$.

In the sequel i and j will always denote positive integers.

LEMMA 10. *Let $\Gamma \in \text{Hom}(A, T)$. If p is odd, or if $m_{j+1} = m_{j+2}$, or if $j \geq 2$ and $m_{j-1} = m_j$, then $\Gamma\gamma_j = \Gamma\delta_j = e$. If $m_j = m_{j+1}$ then $\Gamma\gamma_j = \Gamma\delta_j$.*

Proof. Since $\Gamma \in \text{Hom}(A, T)$ it follows that $\Gamma\sigma^2 = e$ for all $\sigma \in A$.

Suppose p is odd. Then γ_j and δ_j have odd order and hence they are squares. Thus in this case $\Gamma\gamma_j = \Gamma\delta_j = e$.

Next suppose $m_{j+1} = m_{j+2}$. Then $\gamma_j = \sigma_1^2$ and $\delta_j = \sigma_2^2$, where σ_1 and σ_2 are the automorphisms of $D_j \times D_{j+1} \times D_{j+2}$ such that

$$\begin{aligned} \sigma_1 d_j &= d_j d_{j+2}, & \sigma_1 d_{j+1} &= d_{j+1}, & \sigma_1 d_{j+2} &= d_{j+1} d_{j+2}^{-1}, \\ \sigma_2 d_j &= d_j, & \sigma_2 d_{j+1} &= d_{j+1} d_{j+2}, & \sigma_2 d_{j+2} &= d_j^{*j} d_{j+2}^{-1}. \end{aligned}$$

Hence $\Gamma\gamma_j = \Gamma\delta_j = e$.

Now suppose $j \geq 2$ and $m_{j-1} = m_j$. Then $\gamma_j = \sigma_3^2$ and $\delta_j = \sigma_4^2$, where σ_3 and σ_4 are the automorphisms of $D_{j-1} \times D_j \times D_{j+1}$ such that

$$\begin{aligned} \sigma_3 d_{j-1} &= d_{j-1}^{-1} d_{j+1}, & \sigma_3 d_j &= d_{j-1} d_j, & \sigma_3 d_{j+1} &= d_{j+1}, \\ \sigma_4 d_{j-1} &= d_{j-1}^{-1} d_j, & \sigma_4 d_j &= d_j, & \sigma_4 d_{j+1} &= d_{j-1}^{*j} d_{j+1}. \end{aligned}$$

It follows that $\Gamma\gamma_j = \Gamma\delta_j = e$.

Finally suppose $m_j = m_{j+1}$. Then $s_j = 1$ and $\gamma_j\delta_j = \sigma_s^2$, where σ_s is the automorphism of $D_j \times D_{j+1}$ such that

$$\sigma_s d_j = d_{j+1}, \quad \sigma_s d_{j+1} = d_j d_{j+1}.$$

Hence $(\Gamma\gamma_j)(\Gamma\delta_j) = e$ which implies $\Gamma\gamma_j = \Gamma\delta_j$.

LEMMA 11. Let $\Gamma \in \text{Hom}(A, T)$. Let q be an integer relatively prime to p , and let μ_j be the automorphism of $D_j \times D_{j+1}$ such that

$$\mu_j d_j = d_j^q, \quad \mu_j d_{j+1} = d_j^{q-1} d_{j+1}.$$

If $q \equiv 1 \pmod{s_j}$, then $\Gamma\mu_j = e$.

Proof. Suppose $q \equiv 1 \pmod{s_j}$. Then $\mu_j = \nu^2$ where ν is the automorphism of $D_j \times D_{j+1}$ such that

$$\nu d_j = d_j^{-1} d_{j+1}, \quad \nu d_{j+1} = d_j^{q-1} d_{j+1}.$$

Hence $\Gamma\mu_j = (\Gamma\nu)^2 = e$, which establishes Lemma 11.

For p odd let $u(p)$ denote the number of distinct integers in the set $m_1, m_2, \dots, m_l$.

LEMMA 12. If p is odd, then the order of $\text{Hom}(A^{(p)}, T)$ is at most $2^{u(p)}$.

Proof. Since p is an odd prime, the group A_i of automorphisms of D_i is cyclic. Let τ_1 be a generator of A_1 . Then $\tau_1 d_1 = d_1^q$ for some integer q relatively prime to p . For arbitrary i let τ_i be the automorphism of D_i such that $\tau_i d_i = d_i^q$. Then τ_i is a generator of the cyclic group A_i . By Lemma 1, $A^{(p)}$ is generated by the automorphisms $\tau_i, \gamma_j, \delta_j, 1 \leq i \leq l, 1 \leq j < l$. Let $\Gamma \in \text{Hom}(A^{(p)}, T)$. Then $\Gamma\gamma_j = \Gamma\delta_j = e$ for all j by Lemma 10. If $m_i = m_j$ then $\Gamma\tau_i = \Gamma\tau_j$ by Lemma 11. Let $j_1, j_2, \dots, j_{u(p)}$ be a set of indices such that $m_{j_1}, \dots, m_{j_{u(p)}}$ is the complete set of distinct values of m_i . Then Γ is completely determined by the $\Gamma\tau_{j_i}, 1 \leq i \leq u(p)$. Since there are only two possible values for $\Gamma\tau_j$ for each j , the order of $\text{Hom}(A^{(p)}, T)$ is at most $2^{u(p)}$. This completes the proof of Lemma 12.

Let m be one of the numbers $m_1, \dots, m_l$. Thus m is a power of p . Let $D_s, D_{s+1}, \dots, D_t$ be those D_i of order m . For $\sigma \in A$ we write

$$\sigma d_j = \prod_{i=1}^l d_i^{u_{ij}}, \quad s \leq j \leq t.$$

Let $\bar{u}_{ij}$ denote the residue class of u_{ij} modulo p , and let u_σ be the value of the $t-s+1$ rowed determinant $|\bar{u}_{ij}|, s \leq i \leq t, s \leq j \leq t$. Then $\sigma \rightarrow u_\sigma$ is a homomorphism of A onto the multiplicative group of residue classes modulo p . We now define the mapping $\mathfrak{T}_m$ of A into T as follows:

$$\Upsilon_m \sigma = \begin{cases} e & \text{if } u_\sigma \text{ is a square modulo } p, \\ h & \text{otherwise.} \end{cases}$$

Then $\Upsilon_m \in \text{Hom}(A^{(p)}, T)$. If p is odd, then as m runs through its $u(p)$ possible values, Υ_m runs through $u(p)$ independent homomorphisms of A onto T .

Combining this with Lemma 12 we obtain:

LEMMA 13. *If p is an odd prime dividing the order of G , then the order of $\text{Hom}(A^{(p)}, T)$ is $2^{u(p)}$. Furthermore $\text{Hom}(A^{(p)}, T)$ is generated by the $u(p)$ homomorphisms of the form Υ_m , where m is a power of p and an invariant of the finite abelian group G .*

It can be shown that Υ_m depends only on G , T , and m .

We now study $\text{Hom}(A^{(2)}, T)$. We know that $A^{(2)}$ is generated by $\lambda_i, \xi_i, \theta_j, \eta_j, 1 \leq i \leq k, 1 \leq j < k$. We will put $n_0 = r_0 = \infty$ in the following in order to insure that $n_0 > n_1$ and $r_0 \geq 8$.

Let $u(4)$ denote the number of positive values of i such that $r_i \geq 4$, and let $u(8)$ denote the number of positive values of i such that $r_i \geq 8$. Let $u(\eta)$ be the number of values of i such that $n_{i-1} > n_i$ and $n_{i+1} > n_{i+2}$. Let $u(\theta)$ be the number of values of i such that $n_{i-1} > n_i > n_{i+1} > n_{i+2}$. Let $\Gamma \in \text{Hom}(A^{(2)}, T)$.

We note that $\lambda_k = e$ if $n_k = 2$, and that $\xi_k = e$ if $n_k \leq 4$. From Lemma 11 it follows that

$$\Gamma \lambda_i = \Gamma \lambda_{i+1} \quad \text{if } r_i \leq 2$$

and

$$\Gamma \xi_i = \Gamma \xi_{i+1} \quad \text{if } r_i \leq 4.$$

It follows that at most $u(4)$ of the $\Gamma \lambda_i$ are independent and that at most $u(8)$ of the $\Gamma \xi_i$ are independent. From Lemma 10 it follows that $\Gamma \eta_i = \Gamma \theta_i = e$ if either $n_{i+1} = n_{i+2}$ or $n_{i-1} = n_i$. Furthermore $\Gamma \eta_i = \Gamma \theta_i$ if $n_i = n_{i+1}$. It follows that at most $u(\eta)$ of the $\Gamma \eta_i$ are different from e , and that $\Gamma \theta_j$ is independent of $\Gamma \eta_j$ for at most $u(\theta)$ values of j . Thus the order of $\text{Hom}(A^{(2)}, T)$ is at most $2^{u(2)}$, where

$$u(2) = u(4) + u(8) + u(\eta) + u(\theta).$$

We shall now construct $u(2)$ independent homomorphisms of $A^{(2)}$ onto T . For any $\sigma \in A$ we write

$$\sigma c_j = \prod_{i=1}^k c_i^{q_{ij}}, \quad 1 \leq j \leq k.$$

Let $r = 4$ or 8 . Choose s and t such that $s \leq t$, $r_j < r$ if $s \leq j < t$, $r_{s-1} \geq r$, $r_t \geq r$. For each r there are $u(r)$ possible choices of s and t . Let $\bar{q}_{ij}$ denote the residue class of q_{ij} modulo r . Let q_r be the value of the $t-s+1$ rowed deter-

minant $|\bar{q}_{ij}|$, $s \leq i \leq t$, $s \leq j \leq t$. The mapping $\sigma \rightarrow q_\sigma$ is a homomorphism of A onto the multiplicative group of odd residue classes modulo r . For $r=4$ we put

$$\Lambda_t \sigma = \begin{cases} e & \text{if } q_\sigma \equiv 1 \pmod{4}, \\ h & \text{if } q_\sigma \equiv -1 \pmod{4}. \end{cases}$$

For $r=8$ we put

$$\Xi_t \sigma = \begin{cases} e & \text{if } q_\sigma \equiv \pm 1 \pmod{8}, \\ h & \text{if } q_\sigma \equiv \pm 3 \pmod{8}. \end{cases}$$

The $u(4)$ mappings Λ_t and the $u(8)$ mappings Ξ_t are elements of $\text{Hom}(A^{(2)}, T)$. Furthermore

$$\Lambda_t \xi_j = \Lambda_t \theta_j = \Lambda_t \eta_j = e \text{ for all } j,$$

and

$$\Lambda_t \lambda_j = \begin{cases} h & \text{if } s \leq j \leq t, \\ e & \text{otherwise.} \end{cases}$$

Moreover

$$\Xi_t \lambda_j = \Xi_t \theta_j = \Xi_t \eta_j = e \text{ for all } j,$$

and

$$\Xi_t \xi_j = \begin{cases} h & \text{if } s \leq j \leq t, \\ e & \text{otherwise.} \end{cases}$$

Now let s be one of the $u(\eta)$ integers such that $n_{s-1} > n_s$ and $n_{s+1} > n_{s+2}$. Let $\bar{q}_{ij}$ denote the residue class of q_{ij} modulo 2, and let $\mathfrak{M}_\sigma$ be the 2×2 matrix $(\bar{q}_{ij})$, $s \leq i \leq s+1$, $s \leq j \leq s+1$. Then $\sigma \rightarrow \mathfrak{M}_\sigma$ is a homomorphism of A into the group of nonsingular 2×2 matrices over the field of two elements. This group of 2×2 matrices is isomorphic to the symmetric group of order 6 and has an invariant subgroup S of order 3 consisting of

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Put

$$\Psi_s \sigma = \begin{cases} e & \text{if } \mathfrak{M}_\sigma \in S, \\ h & \text{if } \mathfrak{M}_\sigma \notin S. \end{cases}$$

The $u(\eta)$ mappings Ψ_s are elements of $\text{Hom}(A^{(2)}, T)$. For each of them we have

$$\begin{aligned} \Psi_s \lambda_j &= \Psi_s \xi_j = e \text{ for all } j, \\ \Psi_s \theta_j &= \Psi_s \eta_j = e \text{ for } j \neq s, \end{aligned}$$

and $\Psi_s \eta_s = h$, while $\Psi_s \theta_s$ may be either e or h .

Finally let s be any positive integer such that $n_{s-1} > n_s > n_{s+1} > n_{s+2}$. We put

$$\Theta_s \sigma = \begin{cases} e & \text{if } 2r_s \mid q_{s,s+1}, \\ h & \text{otherwise.} \end{cases}$$

Then $\Theta_s \in \text{Hom}(A^{(2)}, T)$. There are $u(\theta)$ such Θ_s and for each one

$$\Theta_s \lambda_j = \Theta_s \xi_j = \Theta_s \eta_j = e \text{ for all } j$$

and

$$\Theta_s \theta_j = \begin{cases} e & \text{if } j \neq s, \\ h & \text{if } j = s. \end{cases}$$

We see that the $u(2)$ homomorphisms $\Lambda_t, \Xi_t, \Psi_s, \Theta_s$ are independent. Therefore the order of $\text{Hom}(A^{(2)}, T)$ is $2^{u(2)}$ and the homomorphisms $\Lambda_t, \Xi_t, \Psi_s, \Theta_s$ form a basis for it.

It can be shown that the homomorphisms $\Lambda_t, \Xi_t, \Psi_s, \Theta_s$ are independent of the decomposition of $G^{(2)}$ into the direct product of cyclic groups C_i and of the choice of generators c_i .

Summarizing the results of this section we obtain:

THEOREM 2. *The order of $\text{Hom}(A, T)$ is 2^u , where $u = \sum u(p)$, the sum being taken over all primes dividing the order of G . There is a basis for $\text{Hom}(A, T)$ consisting of the elements Υ_m , where m runs over the odd prime power invariants of G , Λ_t for those positive t such that $r_t \geq 4$, Ξ_t for those positive t such that $r_t \geq 8$, Ψ_s for those s such that $n_{s-1} > n_s$ and $n_{s+1} > n_{s+2}$, and Θ_s for those s such that $n_{s-1} > n_s > n_{s+1} > n_{s+2}$.*

6. $H^1(A, G)$. In §2 we showed that every nontrivial element of $H^1(A, G)$ has order 2. Therefore $H^1(A, G)$ is either trivial or the direct product of cyclic groups of order 2. It follows that $H^1(A, G)$ is isomorphic to the direct product of the two factor groups

$$Z^1(A, G)/\text{Hom}(A, J)B^1(A, G) \text{ and } \text{Hom}(A, J)B^1(A, G)/B^1(A, G).$$

The first of these two factor groups was determined in §3 and §4, while the latter is isomorphic to $\text{Hom}(A, J)/\text{Hom}(A, J) \cap B^1(A, G)$.

Now if $n_1 = n_2$, then J is trivial and hence

$$H^1(A, G) = Z^1(A, G)/\text{Hom}(A, J)B^1(A, G).$$

If $n_1 > n_2$, then J has order 2 and $\text{Hom}(A, J)$ is given by Theorem 2 with $T = J$. Thus to complete the determination of $H^1(A, G)$ we need only determine the intersection $\text{Hom}(A, J) \cap B^1(A, G)$ under the assumption $n_1 > n_2$.

If $n_1 > n_2$, then we use the notation of §5 with $T = J$ and $h = b_1 = c_1^{r_1/2}$. We note that if Λ_1 is defined, then $r_1 \geq 4$, $\Lambda_1 \sigma = a_1 \sigma a_1^{-1}$ for all $\sigma \in A$, and hence

$$\Lambda_1 = \Gamma_{a_1} \in \text{Hom}(A, J) \cap B^1(A, G).$$

Similarly if Θ_1 is defined, then $n_1 > n_2 > n_3$ and

$$\Theta_1 = \Gamma_{b_2} \in \text{Hom}(A, J) \cap B^1(A, G).$$

Conversely we have:

LEMMA 14. *If Γ is a nontrivial element of $\text{Hom}(A, J) \cap B^1(A, G)$, then Γ is either Λ_1 , Θ_1 , or $\Lambda_1\Theta_1$.*

Proof. If $n_1 = n_2$ then Lemma 14 is trivial. Suppose $n_1 > n_2$ and let a be an element of G such that $\Gamma_a \in \text{Hom}(A, J)$. Then $a\sigma a^{-1} \in J$ for all $\sigma \in A$. Letting σ run through the automorphisms of N_1 we obtain $a \in C_1 \times K_1$. If $a \notin C_1$, then K_1 is not trivial, $n_2 > n_3$, a is of the form $a'b_2$ with $a' \in C_1$, and $\Gamma_a = \Gamma_{a'}\Theta_1$. Hence without loss of generality we may suppose $a \in C_1$. Then $a^2 = a\lambda_1 a^{-1} \in J$ and thus $a^4 = e$. Now $a = c_1^q$ for some integer q . Hence $a\eta_1 a^{-1} = c_2^{-q} \in J \subseteq C_1$. Therefore $n_2 | q$ and $n_1 | r_1 q$. Hence $a^{r_1} = e$. Thus if $a \notin J$, then $a^2 \neq e$, $r_1 \geq 4$, Λ_1 is defined, $aa_1^{-1} \in J$, and $\Gamma_a = \Gamma_{a_1} = \Lambda_1$. On the other hand if $a \in J$, then Γ_a is the identity element of the group $\text{Hom}(A, J)$. Thus Lemma 14 is established.

It now follows that if $n_1 > n_2$, then a set of representatives of a basis of $\text{Hom}(A, J)B^1(A, G)/B^1(A, G)$ can be obtained from the basis of $\text{Hom}(A, J)$ given in Theorem 2 by deleting Λ_1 and Θ_1 (if they are defined). We now sum up in the following theorem.

THEOREM 3. *Let G be a finite abelian group and A its group of automorphisms. Let $\mathcal{S}$ be the set of all v for which Γ_v is defined, $i \leq v \leq iv$. If $n_1 = n_2$, then the cosets $\Gamma_v B^1(A, G)$, $v \in \mathcal{S}$, form a basis of $H^1(A, G)$. If $n_1 > n_2$, then there is a set of representatives of a basis of $H^1(A, G)$ that consists of the elements Γ_v for all $v \in \mathcal{S}$, Υ_m for all odd prime power invariants m of G , Λ_t for all $t > 1$ such that $r_t \geq 4$, Ξ_t for all positive t such that $r_t \geq 8$, Ψ_s for all s such that $n_{s-1} > n_s$ and $n_{s+1} > n_{s+2}$, and Θ_s for all $s > 1$ such that $n_{s-1} > n_s > n_{s+1} > n_{s+2}$. In both cases $H^1(A, G)$ is either trivial or the direct product of groups of order two.*

II. THE OUTER AUTOMORPHISM GROUP OF H

In Part II we will complete the determination of the outer automorphism group of H . Throughout Part II we will use the notation of Part I. In particular H is the holomorph of the finite abelian group G , $\mathcal{O}$ and $\mathcal{J}$ are the outer and inner automorphism groups of H respectively, $\mathcal{A}$ is the group of all automorphisms of H , and $\mathcal{B}$ is the group of all automorphisms of H that map G onto itself.

7. Invariant subgroups of H isomorphic to G . In this section we will collect certain results⁽⁵⁾ about the invariant subgroups of H isomorphic to G .

⁽⁵⁾ Most of these results can be found in [6] in somewhat different form. We will appeal to [6] only for the result that there are no invariant subgroups of H isomorphic to G other than the ones listed in this section.

In particular we will list all these subgroups and, for each such subgroup G^* , we will construct an automorphism of H that maps G onto G^* and leaves $G \cap G^*$ fixed elementwise.

Let A^* be the group of all elements of H of the form (e, σ) , $\sigma \in A$.

LEMMA 15. *Let G^* be an invariant subgroup of H isomorphic to G and suppose that every element of G occurs as the first component of an element of G^* . Let ω be any isomorphism of G onto G^* . For any $g \in G$ let g^* denote the first component of ωg . Then there exists a unique extension Ω of ω to an automorphism of H that maps A^* onto itself. For any $\sigma \in A$ we have $\Omega(e, \sigma) = (e, \sigma^*)$ where*

$$(32) \quad \sigma^* g^* = (\sigma g)^*.$$

Furthermore if Ω' is any automorphism of H that maps G onto G^ and A^* onto itself, then $\Omega'^{-1}\Omega' \in \mathcal{G}$. If Ω^* is any automorphism of H that maps both G and A^* onto themselves, then $\Omega^* \in \mathcal{G}$.*

Proof. Since G and G^* are finite and have the same order it follows that every element of G occurs exactly once as the first component of an element of G^* . Hence the mapping $g \rightarrow g^*$ is a one-to-one mapping of G onto itself. Suppose that Ω is an automorphism of H mapping A^* onto itself such that ω is the restriction of Ω to G . For any $\sigma \in A$ we put $\Omega(e, \sigma) = (e, \sigma^*)$. Now

$$\Omega \sigma g = \Omega \{ (e, \sigma) g (e, \sigma)^{-1} \} = (e, \sigma^*) (g^*, \phi_g) (e, \sigma^*)^{-1}$$

where $\omega g = (g^*, \phi_g)$. Comparing first components we obtain (32), which determines σ^* uniquely. Since Ω is completely determined by its effect on G and A^* it follows that there is at most one automorphism of H with the required properties.

We must now prove the existence of Ω . For any $\sigma \in A$ let $[\sigma]$ denote the automorphism of G^* induced by the inner automorphism $I_{(e, \sigma)}$ of H . Since every element of G occurs as the first component of an element of G^* it follows that the mapping $\sigma \rightarrow [\sigma]$ is one-to-one of A into the group of automorphisms of G^* . Since A and the group of automorphisms of G^* have the same finite order it follows that $\sigma \rightarrow [\sigma]$ is an isomorphism of A onto this group of automorphisms. Let σ^* be the automorphism of G such that $[\sigma^*] = \omega \sigma \omega^{-1}$. Then the mapping Ω given by

$$\Omega(g, \sigma) = \omega g (e, \sigma^*)$$

is a one-to-one mapping of H onto itself. Clearly $\Omega g = \omega g$ for all $g \in G$ and Ω maps A^* onto itself. Furthermore

$$\begin{aligned} \{ \Omega(a, \sigma) \} \{ \Omega(b, \tau) \} &= \omega a \{ [\sigma^*] \omega b \} (e, \sigma^* \tau^*) \\ &= \omega (a \sigma b) (e, \sigma^* \tau^*) \\ &= \Omega \{ (a, \sigma) (b, \tau) \}. \end{aligned}$$

Hence Ω is an automorphism with the desired properties.

Now Ω' maps G onto G^* . Hence for suitable $\phi \in A$ we have $\Omega' \phi g = \omega g$ for all $g \in G$. Then $\Omega' I_{(\epsilon, \phi)}$ is an extension of ω to an automorphism of H that maps A^* onto itself. Hence $\Omega' I_{(\epsilon, \phi)} = \Omega$ by the uniqueness of Ω . Thus $\Omega^{-1} \Omega' \in \mathcal{J}$. Finally putting $G^* = G$, $\Omega = I$, the identity automorphism of H , and $\Omega' = \Omega^*$ this becomes $\Omega^* \in \mathcal{J}$, which completes the proof.

If $n_1 \geq 4$ let ϕ_1 denote the automorphism of $G^{(2)}$ such that

$$\phi_1 g = g^{1+n_1/2}$$

for all $g \in G^{(2)}$. Clearly ϕ_1 belongs to the center of A and $\phi_1^2 = \epsilon$.

If $n_2 \geq 2$ put $\phi_2 = \theta_1^{n_2/2}$. Thus

$$\phi_2 c_1 = c_1, \quad \phi_2 c_2 = c_1^{n_1/2} c_2.$$

Furthermore $\phi_2^2 = \epsilon$.

If $n_1 \geq 4$ put $\phi_3 = \phi_1 \eta_1^{n_1/2}$. Thus

$$\phi_3 c_1 = c_1^{1+n_1/2} c_2^{n_1/2}, \quad \phi_3 c_2 = c_2^{1+n_1/2}$$

We have $\phi_3^2 = \epsilon$.

We will now list the invariant subgroups of H isomorphic to G .

1. Suppose $n_1 \geq 8$ and $n_1 > n_2$. In this case let G_1 be the group generated by (c_1, ϕ_1) and N_1 . Then G_1 is isomorphic to G . The group G_1 consists of those elements of H of the form (a, ϕ_1) where n_1 divides the order of a and of those elements of G whose order is not divisible by n_1 . It follows that G_1 is an invariant subgroup of H isomorphic to G , that every element of G occurs exactly once as the first component of an element of G_1 , and that $G \cap G_1$ is generated by c_1^2 and N_1 . Let ω_1 be the isomorphism of G onto G_1 such that

$$\omega_1 c_1 = (c_1^{1+n_1/4}, \phi_1)$$

and $\omega_1 g = g$ for all $g \in N_1$. Then ω_1 leaves every element of $G \cap G_1$ fixed. By Lemma 15 there exists an automorphism Ω_1 of H that extends ω_1 and that maps A^* onto itself. By (32) we have

$$(33) \quad \begin{cases} \Omega_1(e, \sigma) = (e, \sigma) \text{ if } \sigma c_1 = c_1, \\ \Omega_1(e, \xi_1) = (e, \xi_1), \\ \Omega_1(e, \lambda_1) = (e, \lambda_1 \phi_1), \\ \Omega_1(e, \eta_1) = (e, \eta_1^{1+n_1/4}). \end{cases}$$

Since $n_1 \geq 8$ we have $\phi_1 = \xi_1^{n_1/8}$. Hence $\Omega_1(e, \phi_1) = (e, \phi_1)$. It follows that Ω_1^2 maps both G and A^* onto themselves. Hence $\Omega_1^2 \in \mathcal{J}$ by the last statement of Lemma 15.

2. Suppose $n_1 > n_2 > n_3$. In this case let G_2 be the group generated by (c_1, ϕ_2) , (c_2, ϕ_1) , and $N_{1,2}$. Then G_2 is isomorphic to G and $G \cap G_2$ is the characteristic subgroup of G generated by c_1^2 , c_2^2 , and $N_{1,2}$. Let F be the set of all elements of G that do not belong to G_2 and have order divisible by n_2 but not by

n_1 . For any element a of G whose order is divisible by n_1 let ϕ_a be the unique automorphism of G such that $\phi_a a = a$ and $\phi_a f = hf$ for all $f \in F$, where h is the nonidentity element of J . The group G_2 consists of the elements of $G \cap G_2$, the elements of H of the form (f, ϕ_1) , $f \in F$, and the elements of H of the form (a, ϕ_a) where n_1 divides the order of a . It follows that G_2 is an invariant subgroup of H . Thus G_2 is an invariant subgroup of H isomorphic to G and every element of G occurs exactly once as the first component of an element of G_2 . Let ω_2 be the isomorphism of G onto G_2 such that

$$\omega_2 c_1 = (c_1, \phi_2), \quad \omega_2 c_2 = (c_2, \phi_1),$$

and $\omega_2 g = g$ for all $g \in N_{1,2}$. Then ω_2 leaves every element of $G \cap G_2$ fixed. Let Ω_2 be the automorphism of H that extends ω_2 and maps A^* onto itself. By (32) we have

$$(34) \quad \begin{cases} \Omega_2(e, \sigma) = (e, \sigma) \text{ if } \sigma c_1 \in C_1, \\ \Omega_2(e, \eta_1) = (e, \eta_1 \phi_1). \end{cases}$$

It follows from (34) that $\Omega_2(e, \phi_1) = (e, \phi_1)$ and $\Omega_2(e, \phi_2) = (e, \phi_2)$. Therefore Ω_2^2 acts as the identity on G . Since Ω_2^2 maps A^* onto itself it follows from Lemma 15, with $G^* = G$, that $\Omega_2^2 = I$, the identity automorphism of H .

3. Suppose $n_1 \geq 8$ and $n_2 > n_3$. In this case let G_3 be the group generated by $(c_1, \phi_1 \phi_2)$, (c_2, ϕ_3) , and $N_{1,2}$. Then G_3 is isomorphic to G and every element of G occurs exactly once as the first component of an element of G_3 . Furthermore $G \cap G_3$ is the characteristic subgroup of G generated by c_1^2 , c_2^2 , and $N_{1,2}$. We now consider two cases:

3a. Suppose $n_1 \geq 8$ and $n_1 > n_2 > n_3$. In this case G_1 , G_2 , Ω_1 , and Ω_2 are defined. Moreover $\phi_3 = \phi_1$ and

$$G \cap G_3 = G \cap G_2 \subset G \cap G_1.$$

Hence $\Omega_1 \Omega_2$ and $\Omega_2 \Omega_1$ both leave $G \cap G_3$ fixed elementwise. Furthermore

$$\Omega_1 \Omega_2 c_1 = (c_1^{1+n_1/4}, \phi_1 \phi_2) = \Omega_2 \Omega_1 c_1$$

and

$$\Omega_1 \Omega_2 c_2 = (c_2, \phi_1) = \Omega_2 \Omega_1 c_2.$$

It follows that $\Omega_1 \Omega_2$ and $\Omega_2 \Omega_1$ map G onto G_3 , that their restrictions to G are identical, and that they both map A^* onto itself. Since $\Omega_1 \Omega_2$ is an automorphism of H it follows that G_3 is an invariant subgroup of H isomorphic to G . Hence $\Omega_1 \Omega_2 = \Omega_2 \Omega_1$ by Lemma 15. In this case we put $\Omega_3 = \Omega_1 \Omega_2$.

3b. Suppose $n_1 \geq 8$ and $n_1 = n_2 > n_3$. For any $a \in G$ whose order n_a is divisible by n_1 let θ_a be the unique automorphism of G such that $\theta_a a = a$, $\theta_a g' = g'$ for all $g' \in G'$, and $\theta_a g = a^{n_a/2} g$ for every $g \in G$ of order n_1 that is independent of a .

Then G_3 consists of the elements of $G \cap G_3$ and the elements of H of the form $(a, \theta_a \phi_1)$ where n_1 divides the order of a . It follows that G_3 is an invariant subgroup of H isomorphic to G . Let ω_3 be the isomorphism of G onto G_3 such that

$$\omega_3 c_1 = (c_1^{1+n_1/4}, \phi_1 \phi_2), \quad \omega_3 c_2 = (c_2^{1+n_1/4}, \phi_3),$$

and $\omega_3 g = g$ for all $g \in N_{1,2}$. Then ω_3 leaves $G \cap G_3$ fixed elementwise. Let Ω_3 be the automorphism of H that extends ω_3 and maps A^* onto itself. By (32) we have $\Omega_3(e, \phi_i) = (e, \phi_i)$, $i = 1, 2, 3$. It follows that Ω_3^2 maps both G and A^* onto themselves. Hence $\Omega_3^2 \in \mathcal{G}$ by Lemma 15. In this case we do not need to know $\Omega_3(e, \sigma)$ for all $\sigma \in A$. However if g^* denotes the first component of $\Omega_3 g$, then $g^* g^{-1}$ is a square. Therefore if σ^* is the automorphisms of G such that $\Omega_3(e, \sigma) = (e, \sigma^*)$ and if

$$\sigma c_j = \prod_{i=1}^k c_i^{q_{ij}}, \quad \sigma^* c_j = \prod_{i=1}^k c_i^{q_{ij}^*}, \quad 1 \leq j \leq k,$$

then it follows from (32) that

$$(35) \quad q_{ij} = q_{ij}^* \pmod{2}.$$

4. Suppose $n_1 = 4$ and $n_2 = k = 2$. Then Γ_i and Ω_2 are both defined, and Γ_i can be regarded as an automorphism of H by (2). Referring to §4 we have $a_1 = c_1$, $b_2 = c_2$, $\Gamma_i = \Gamma'$,

$$\begin{aligned} \Gamma_i \Omega_2 c_1^{-1} &= \Gamma_i(c_1^{-1}, \phi_2) = \Gamma_i(c_1^{-1}, \theta_1) = (c_2, \theta_1), \\ \Gamma_i \Omega_2 c_2 &= \Gamma_i(c_2, \phi_1) = \Gamma_i(c_2, \lambda_1) = (e, \lambda_1), \end{aligned}$$

and $\Gamma_i \Omega_2 g' = g'$ for all $g' \in G' = N_{1,2}$. Since Γ_i and Ω_2 are automorphisms of H , it follows that (c_2, θ_1) , (e, λ_1) , and G' generate an invariant subgroup of H isomorphic to G . We designate this subgroup by G_4 . The automorphism $\Gamma_i \Omega_2$ of H maps G onto G_4 .

Conversely it has been shown [6] that if G^* is an invariant subgroup of H isomorphic to G , and $G^* \neq G$, then either

1. $n_1 \geq 8$, $n_1 > n_2$, and $G^* = G_1$, or
2. $n_1 > n_2 > n_3$ and $G^* = G_2$, or
3. $n_1 \geq 8$, $n_2 > n_3$, and $G^* = G_3$, or
4. $n_1 = 4$, $n_2 = k = 2$, and $G^* = G_4$.

For each of the groups G_w , $1 \leq w \leq 3$, we have an automorphism Ω_w of H that maps G onto G_w , leaves $G \cap G_w$ fixed elementwise, maps A^* onto itself, and satisfies $\Omega_w^2 \in \mathcal{G}$. Furthermore Ω_1 is defined if the first two of the inequalities

$$(36) \quad n_1 \geq 8, \quad n_1 > n_2, \quad n_2 > n_3$$

hold, Ω_2 is defined if the last two of the inequalities (36) hold, and Ω_3 is defined if the first and third of these inequalities hold. Therefore if more than one of

these Ω_w is defined, then all three of them are and we have $\Omega_3 = \Omega_1\Omega_2 = \Omega_2\Omega_1$.

Let $\mathcal{O}$ be the subgroup of $\mathcal{O}$ that is generated by the cosets $\Omega_w\mathcal{G}$, where w runs over those values for which Ω_w is defined, $1 \leq w \leq 3$. If all three of the inequalities (36) hold, then $\mathcal{O}$ is the four group. If exactly two of these inequalities hold, then $\mathcal{O}$ has order 2. If less than two of them hold, then $\mathcal{O}$ is trivial.

8. The outer automorphism group of H . The group G is an invariant subgroup of H . Hence any automorphism Ω of H maps G onto a group G^* , where G^* is an invariant subgroup of H isomorphic to G . As indicated in §7 there are five possibilities for G^* , namely G , G_1 , G_2 , G_3 , and G_4 . If $G^* = G$, then $\Omega \in \mathcal{B}$. If $G^* = G_w$ with $w = 1, 2$, or 3 , then $\Omega_w^{-1}\Omega \in \mathcal{B}$. If $G^* = G_4$, then $(\Gamma_1\Omega_2)^{-1}\Omega \in \mathcal{B}$ and $\Gamma_1 \in Z^1(A, G) \subseteq \mathcal{B}$. It follows that $\mathcal{A}$ is generated by $\mathcal{B}$ and those Ω_w that are defined, $1 \leq w \leq 3$.

In §1 we showed that $H^1(A, G)$ is isomorphic to $\mathcal{B}/\mathcal{G}$ under the natural isomorphism $\Gamma B^1(A, G) \rightarrow \Gamma\mathcal{G}$. In this section we will identify $H^1(A, G)$ and $\mathcal{B}/\mathcal{G}$ by means of this isomorphism. Then $\mathcal{O}$ is generated by $\mathcal{O}$ and $H^1(A, G)$, and $\mathcal{O} \cap H^1(A, G)$ consists of the identity alone. We must now study the relations between the elements of $\mathcal{O}$ and those of $H^1(A, G)$. In other words we must study the relations, modulo $\mathcal{G}$, between the Ω_w and the cocycles of $Z^1(A, G)$.

Henceforth w will denote an integer such that Ω_w is defined, $1 \leq w \leq 3$. For any $\sigma \in A$, σ^* will denote the element of A such that $\Omega_w(e, \sigma) = (e, \sigma^*)$, and for any $g \in G$, g^* will denote the first component of $\Omega_w g$. This is in agreement with the notation of Lemma 15.

LEMMA 16. *If Ω_w is defined and $\Gamma \in \text{Hom}(A, J)$, then*

$$\Gamma\Omega_w \equiv \Omega_w\Gamma \pmod{\mathcal{G}}.$$

Proof. If $n_1 = n_2$, then J is trivial and so is Lemma 16. Hence suppose $n_1 > n_2$. Then if Ω_3 is defined we have $\Omega_3 = \Omega_1\Omega_2$. Thus without loss of generality suppose that $w = 1$ or 2 . Since $J \subset G_w$ it follows that Γ maps G_w onto itself. Now $\Omega_w^2 \in \mathcal{G}$ and hence Ω_w permutes G and G_w . Therefore $(\Omega_w\Gamma)^2$ maps G onto itself. For all $\sigma \in A$ we have

$$(37) \quad (\Omega_w\Gamma)^2(e, \sigma) = ((\Gamma\sigma)(\Gamma\sigma^*), \sigma^{**}),$$

since $\Gamma\sigma$ and $\Gamma\sigma^*$ are elements of J and hence left fixed by Ω_w . We distinguish two cases:

Case 1. $\Gamma\phi_1 = e$. If $w = 1$, then $n_1 \geq 8$, $\eta_1^{n_1/4}$ is a square, and it follows from (33) that $\Gamma\sigma = \Gamma\sigma^*$ for all $\sigma \in A$. If $w = 2$ it follows from (34) that $\Gamma\sigma = \Gamma\sigma^*$ for all $\sigma \in A$. In either case $(\Omega_w\Gamma)^2$ maps A^* onto itself by (37). It now follows from Lemma 15 that $(\Omega_w\Gamma)^2 \in \mathcal{G}$ which is equivalent to $\Gamma\Omega_w \equiv \Omega_w\Gamma \pmod{\mathcal{G}}$.

Case 2. $\Gamma\phi_1 \neq e$. Here $\Gamma\phi_1 = c_1^{n_1/2}$. If $n_2 \geq 2$, then $\Gamma\phi_1 = e$ by Lemma 11 with $q = 1 + n_1/2$ and $j = 1$. Hence $n_2 = 1$. Therefore Ω_2 is undefined, $w = 1$, and $n_1 \geq 8$. By (33) we have

$$(\Gamma\lambda_1)(\Gamma\lambda_1^*) = (\Gamma\lambda_1)^2(\Gamma\phi_1) = \Gamma\phi_1 = c_1^{n_1/2},$$

and

$$(\Gamma\sigma)(\Gamma\sigma^*) = (\Gamma\sigma)^2 = e$$

for all $\sigma \in A'$ and for $\sigma = \xi_1$. Since $n_2 = 1$ it follows that A is generated by λ_1 , ξ_1 , and A' . Therefore $(\Omega_w\Gamma)^2 I_{a_1}$ maps A^* onto itself, where $a_1 = c_1^{n_1/4}$ as usual. Clearly I_{a_1} maps G onto itself. Therefore $(\Omega_w\Gamma)^2 I_{a_1}$ also maps G onto itself, and Lemma 15 yields $(\Omega_w\Gamma)^2 I_{a_1} \in \mathcal{G}$. Hence $(\Omega_w\Gamma)^2 \in \mathcal{G}$ and Lemma 16 is established.

We now come to the relations between the special cocycles Γ_v of §4 and the Ω_w . The key to these relations is the following:

LEMMA 17. *Suppose that Γ_v and Ω_w are defined. Let $\bar{G}$ be the characteristic subgroup of G used to define Γ_v . Then we have $n_1 \geq 8$ and $\bar{G} \subseteq G \cap G_w$ with the exception of the case $n_1 > n_2 = k = 2$, $w \neq 1$, $v = i$.*

Proof. We recall that if $v = i$, then either $n_1 > n_2 > n_3$ and $\bar{G}$ is generated by a_1 and b_2 , or $n_1 = n_2 \geq 4n_3$ and $\bar{G}$ is generated by a_1 and a_2 . If $v = ii$ or iii , then $n_3 > n_4$ and $\bar{G}$ is generated by b_1 , b_2 , and b_3 . If $v = iv$ then $n_1 = 4n_2$ and $\bar{G}$ is generated by a_1 .

Suppose that $n_1 \leq 4$. Then $w = 2$ and $n_1 > n_2 > n_3$. This implies $n_1 = 4$, $n_2 = k = 2$, and $v = i$. Thus $n_1 \leq 4$ can occur only in the exceptional case.

Now suppose that $n_1 \geq 8$. If $n_2 = 1$, then none of the Γ_v are defined. Therefore $n_2 \geq 2$. Now c_1^2 , c_2^2 , and c_3 are elements of G_w . Hence a_1 and b_1 are elements of G_w . Furthermore if $n_3 > n_4$, then $n_3 \geq 2$ and $b_3 \in G_w$. If $n_1 = n_2$, then $a_2 \in G_w$.

Now suppose that $n_1 \geq 8$ and that $\bar{G} \not\subseteq G \cap G_w$. Since $\bar{G}$ is a subgroup of G we have $\bar{G} \not\subseteq G_w$. Then $v \neq iv$, $n_2 \geq 2$, and $b_2 \notin G_w$. If $n_2 \geq 4$, then $b_2 \in G_w$. Hence $n_2 = 2$. If $w = 1$, then $b_2 = c_2 \in G_w$. Hence $w \neq 1$. If $n_2 = n_3$, then $w = 1$. Therefore $n_2 > n_3$ and we have $n_3 = 1$, $k = 2$, and $v \neq ii, iii$. Thus $v = i$ and we have the exceptional case mentioned in the statement of Lemma 17.

LEMMA 18. *Suppose that Γ_v and Ω_w are defined and that the exceptional case of Lemma 17 does not hold. Let $\sigma \in A$ and $(a, \phi) \in G_w$. Then $\Omega_w \Gamma_v \sigma = \Gamma_v \sigma$, $\Gamma_v \sigma^* = \Gamma_v \sigma$, and $\Gamma_v \phi = e$.*

Proof. By Lemma 17 we have $n_1 \geq 8$ and $\bar{G} \subseteq G \cap G_w$. It follows from the construction of Γ_v that $\Gamma_v \sigma \in \bar{G}$. Since Ω_w leaves every element of $G \cap G_w$ fixed we have $\Omega_w \Gamma_v \sigma = \Gamma_v \sigma$. To prove the remaining statements of Lemma 18 we distinguish two cases.

Case 1. $v = iii$. Here $n_1 = n_2$ so that $w = 3$. We recall that $\Gamma_{iii}\sigma$ depends only on the residue classes of the numbers q_{ij} modulo 2, where

$$\sigma c_j = \prod c_i^{q_{ij}}.$$

Hence $\Gamma_{iii}\phi_j = e$ for those ϕ_j that are defined, $1 \leq j \leq 3$. Now ϕ is contained in

the group generated by these ϕ_j . Hence $\Gamma_{iii}\phi = e$. Furthermore $\Gamma_{iii}\sigma^* = \Gamma_{iii}\sigma$ by (35).

Case 2. $v \neq iii$. In this case $\Gamma_v\sigma$ depends only on the restriction of σ to $\bar{G}$. Since G_w is abelian it follows that ϕ leaves every element of $G \cap G_w$ fixed. Therefore ϕ leaves every element of $\bar{G}$ fixed and we have $\Gamma_v\phi = e$. Now $\Omega_w\bar{g} = \bar{g}$ for all $\bar{g} \in \bar{G}$ and $\bar{G}$ is a characteristic subgroup of G . Hence, by (32) we have

$$\sigma^*\bar{g} = \sigma^*\bar{g}^* = (\sigma\bar{g})^* = \sigma\bar{g}$$

for all $\bar{g} \in \bar{G}$. Therefore $\Gamma_v\sigma^* = \Gamma_v\sigma$ and Lemma 18 is established.

LEMMA 19. *Suppose that Γ_v and Ω_w are defined. Then $\Gamma_v\Omega_w = \Omega_w\Gamma_v$ with the exception of the case $n_1 > n_2 = k = 2$, $w \neq 1$, $v = i$.*

Proof. Suppose that the exceptional case $n_1 > n_2 = k = 2$, $w \neq 1$, $v = i$ does not hold. Then Lemma 18 applies. Hence for any $g \in G$ we have

$$\Gamma_v\Omega_w g = \Gamma_v(g^*, \phi) = (g^*, \phi) = \Omega_w g = \Omega_w\Gamma_v g$$

for suitable $\phi \in A$. Applying Lemma 18 again we have, for any $\sigma \in A$,

$$\Gamma_v\Omega_w(e, \sigma) = (\Gamma_v\sigma^*, \sigma^*) = (\Gamma_v\sigma, \sigma^*)$$

and

$$\Omega_w\Gamma_v(e, \sigma) = \Omega_w(\Gamma_v\sigma, \sigma) = (\Gamma_v\sigma, \sigma^*).$$

Therefore $\Gamma_v\Omega_w = \Omega_w\Gamma_v$ and Lemma 19 is established. We will see that Ω_w and Γ_v do not commute in the exceptional case.

It follows from Lemmas 16 and 19 that, except for the case $n_1 > n_2 = k = 2$, the elements of $\mathcal{O}$ and $H^1(A, G)$ commute with each other. Thus $\mathcal{O} = \mathcal{O} \times H^1(A, G)$ except for this exceptional case.

We now come to the case $n_1 > n_2 = k = 2$. The existence of G_4 , if $n_1 = 4$, indicates that the cases $n_1 \geq 8$ and $n_1 = 4$ behave differently. We treat them separately.

Suppose $n_1 \geq 8$ and $n_2 = k = 2$. In this case we need to determine $(\Omega_2\Gamma_i)^2$ modulo $\mathcal{J}$. We see that $\Omega_2\Gamma_i$ maps G onto G_2 . Since G , G_1 , G_2 , and G_3 are the only invariant subgroups of H isomorphic to G it follows that $\Omega_2\Gamma_i$ maps G_2 onto one of these groups. Now $(c_1, \phi_2) = (c_1, \theta_1) \in G_2$ and

$$\Omega_2\Gamma_i(c_1, \theta_1) = (c_1^{1+3n_1/4}, c_2, \phi_1)$$

which is an element of G_1 , but not of G , G_2 , or G_3 . Hence $\Omega_2\Gamma_i$ maps G_2 onto G_1 , and $(\Omega_2\Gamma_i)^2$ maps G onto G_1 . Since $n_2 = k = 2$ it follows that A is generated by λ_1 , ξ_1 , θ_1 , η_1 , and the elements of A' . We have

$$\begin{aligned} (\Omega_2\Gamma_i)^2(e, \lambda_1) &= (e, \phi_1\lambda_1), \\ (\Omega_2\Gamma_i)^2(e, \theta_1) &= (c_1^{n_1/2}, \phi_1\theta_1), \end{aligned}$$

and

$$(\Omega_2 \Gamma_i)^2(e, \sigma) = (e, \sigma)$$

for all $\sigma \in A'$, for $\sigma = \xi_1$, and for $\sigma = \eta_1$. Hence $I_{c_2}(\Omega_2 \Gamma_i)^2$ maps A^* onto itself. Since G_1 is an invariant subgroup of H it follows that I_{c_2} maps G_1 onto itself, and hence $I_{c_2}(\Omega_2 \Gamma_i)^2$ maps G onto G_1 . Therefore $I_{c_2}(\Omega_2 \Gamma_i)^2 \equiv \Omega_1 \pmod{\mathfrak{g}}$ by Lemma 15. Thus we have $(\Omega_2 \Gamma_i)^2 \equiv \Omega_1 \pmod{\mathfrak{g}}$. We see at once that the cosets $\Gamma_i \mathfrak{g}$ and $\Omega_2 \mathfrak{g}$ generate a group Θ_8 of order 8, and that Θ_8 is the octic group. Since $k=2$, neither Γ_{ii} nor Γ_{iii} is defined. Furthermore Γ_{iv} is defined if $n_1=8$ and undefined if $n_1 \geq 16$. If $n_1 \geq 16$ we put $\Theta' = \text{Hom}(A, J)\mathfrak{g}/\mathfrak{g}$, while if $n_1=8$ we let Θ' be the group obtained from $\text{Hom}(A, J)\mathfrak{g}/\mathfrak{g}$ by adjoining $\Gamma_{iv}\mathfrak{g}$. Then Θ' is the direct product of groups of order 2, and $\Theta = \Theta_8 \times \Theta'$. Referring to Lemma 14 we see that a set of representatives of a basis of $\text{Hom}(A, J)\mathfrak{g}/\mathfrak{g}$ can be obtained by deleting Λ_1 and Θ_1 from the basis of $\text{Hom}(A, J)$ given by Theorem 2.

There remains only the case $n_1=4$, $n_2=k=2$ to be treated. In this case the invariant subgroups of H isomorphic to G are G , G_2 , and G_4 . We observed in §7 that $\Gamma_i \Omega_2$ maps G onto G_4 . Furthermore Ω_2 maps G_2 onto G so that $\Gamma_i \Omega_2$ also maps G_2 onto G . It follows immediately that $\Gamma_i \Omega_2$ maps G_4 onto G_2 , and that $(\Gamma_i \Omega_2)^3$ maps G onto itself. In this case A is generated by λ_1 , θ_1 , η_1 , and the elements of A' . We have

$$(\Gamma_i \Omega_2)^3(e, \theta_1^2) = (c_1, \theta_1 \lambda_1),$$

and

$$(\Gamma_i \Omega_2)^3(e, \sigma) = (e, \sigma)$$

for all $\sigma \in A'$, for $\sigma = \lambda_1$, and for $\sigma = \eta_1$. It follows that $I_{c_2}(\Gamma_i \Omega_2)^3$ maps both G and A^* onto themselves. Hence $I_{c_2}(\Gamma_i \Omega_2)^3 \in \mathfrak{g}$ by Lemma 15. Therefore $(\Gamma_i \Omega_2)^3 \in \mathfrak{g}$ and we see that the cosets $\Gamma_i \mathfrak{g}$ and $\Omega_2 \mathfrak{g}$ generate a group Θ_6 isomorphic to the symmetric group of order 6. In fact if we regard Θ_6 as a permutation group on G , G_2 , G_4 , then it is exactly this symmetric group. In this case Γ_{ii} , Γ_{iii} , Γ_{iv} , Ω_1 , and Ω_3 are undefined and we have

$$\Theta = \Theta_6 \times \text{Hom}(A, J)\mathfrak{g}/\mathfrak{g}.$$

In this case Λ_1 is undefined and we can obtain a set of representatives of a basis of $\text{Hom}(A, J)\mathfrak{g}/\mathfrak{g}$ by deleting Θ_1 from the basis of $\text{Hom}(A, J)$ given by Theorem 2. This set consists of Ψ_1 and the Υ_m , where m runs through the odd prime power invariants of G .

We summarize our results in a final theorem:

THEOREM 4. *If G is the direct product of a cyclic group of order four, a group of order two, and an abelian group of odd order, then Θ is the direct product of the symmetric group of order six and a finite number of groups of order two, $\Theta = \Theta_6 \times \text{Hom}(A, J)\mathfrak{g}/\mathfrak{g}$. If G is the direct product of a cyclic group of order 2^n ,*

$n \geq 3$, a group of order two, and an abelian group of odd order, then Θ is the direct product of the octic group and a finite number of groups of order two, $\Theta = \Theta_8 \times \Theta'$. For all other finite abelian groups G , Θ is the direct product of a finite number of groups of order two, in fact $\Theta = \mathcal{O} \times H^1(A, G)$.

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